Methods of Differential Geometry in Classical Field Theories

k-Symplectic and k-Cosymplectic Approaches

Authors: Manuel de León, Modesto Salgado, Silvia Vilariño The classical field theory has evolved from the mechanics of the continuum, which is usually treat as the natural byproduct of the classical mechanics.So the book begins with an introductory chapter to mechanics with a dual exposition: one with the cotangent fiber to the manifold of generalized coordinates Q, T^*Q, which nowadays it's called hamiltonian mechanics; and the other one with the tangent fiber TQ which is the lagrangian mechanics. The chapter ends by presenting the relationship between the two expositions through the Legendre transformation.

Now, the book forks to show two differents generalizations of classical mechanics to describe the classical field theory: the $k-symplectic$ formalism, and the field theory that involves the independent parameters, i.e. the "space-time" coordinates: the $k-cosymplectic$ formalism..Both of them generalizes then the hamiltonian and lagrangian mechanics, and each generalization have a double way to get the field equations. The first one is through a variational principle over a certain manifold, and the second one is through the flow of a vector field defined in the same manifold and with the help of certain canonical differential forms over that manifold.

I think the book can be resumed with the following formulas:

1. k symplectic formalism

   (a) Hamiltonian field equations

   The manifold M they use is k copies of the cotangent fiber, which can also be describe by the fiber of jets of functions from the generalized coordinates Q (of dimension n and coordinates q^i) that goes to zero over $\mathbb{R}^k$:

$$M = \overset{k}{\oplus} T^*Q = J^1(Q, \mathbb{R}^k)_0$$

   with coordinates (q^i, p_i^α).This manifold have k Liouville forms $\theta^\alpha = p_i^\alpha dq^i$ which give rise to k forms, $\omega^\alpha = -d\theta^\alpha$ which are not symplectic over M, (they are symplectic on their respective copy of T^*Q) but verifies:

$$\begin{cases} \overset{\alpha}{\cap}\, Ker\ \omega^\alpha = 0 \\ \omega^\alpha|_{V\times V} = 0 \quad \forall\alpha \end{cases}$$

   where V is an integrable distribution on M of dimension nk.

   With a hamiltonian $H : M \to \mathbb{R}$ the authors develop the theory from the point of view of the variational calculus and from the differential geometry.

i. Variational calculus
The action they consider is

$$\begin{array}{rccc} \mathcal{A}: & C_c^\infty(\mathbb{R}^k, M) & \longrightarrow & \mathbb{R} \\ & \varphi & & \displaystyle\int_{\mathbb{R}^k} \varphi^*\theta^\alpha \wedge d^{k-1}x^\alpha - (\varphi^* H) \; d^k x^\alpha \end{array}$$

where $d^k x^\alpha$ is the $k-$volume form and $d^{k-1}x^\alpha = dx^1 \wedge \ldots \wedge \widehat{dx^\alpha} \wedge \ldots \wedge dx^k$. The field equations are derived from:

$$\delta \mathcal{A} = 0$$

ii. Geometrical settings
The most peculiar ingredient of the book is what they call a $k-$vector field, $X \in \mathfrak{X}^k(M)$ because it isn't a vector field on M but rather k vector fields on M, X_α written together. An integral section of $X \in \mathfrak{X}^k(M)$ passing through a point $p \in M$, consist of a map $\varphi : U_0 \subset \mathbb{R}^k \to M$, defined in some neighborhood U_0 of $0 \in \mathbb{R}^k$ such that

$$\varphi(0) = p \quad \varphi_*(x) \left(\partial_{x^\alpha}|_x \right) = X_\alpha(\varphi(x))$$

The field equations come from the flow of the $k-$vector field that verify:

$$i_{X_\alpha} \omega^\alpha = dH$$

In any case the field equations are written for

$$\begin{array}{rccc} \varphi: & \mathbb{R}^k & \longrightarrow & M \\ & x & & \left(\psi^i(x), \psi_i^\alpha(x)\right) \end{array}$$

and takes the form

$$\begin{cases} \frac{\partial H}{\partial q^i}(\varphi(x)) = -\displaystyle\sum_{\alpha=1}^{k} \frac{\partial \psi_i^\alpha}{\partial x^\alpha}(x) \\ \\ \frac{\partial H}{\partial p_i^\alpha}(\varphi(x)) = \frac{\partial \psi^i}{\partial x^\alpha}(x) \end{cases}$$

(b) Lagrangian field equations
The manifold N is k copies of the tangent fiber, which can also be describe by the fiber of jets of functions from $\mathbb{R}^k$ to the manifold of the generalized coordinates Q (of dimension n and coordinates q^i) evaluated at $0 \in \mathbb{R}^k$:

$$N = \overset{k}{\oplus} TQ = J_0^1(\mathbb{R}^k, Q)$$

with coordinates (q^i, v^i_α) and projection $\tau^k : N \to Q$. This manifold has a canonical family of k tensor fields $J^\alpha \in \mathfrak{X}^1_1(N)$ define in coordinates as:

$$J^\alpha = \partial_{v^i_\alpha} \otimes dq^i \tag{1}$$

and also the Liouville vector field $\Delta \in \mathfrak{X}(N)$ define as:

$$\Delta = v^i_\alpha \ \partial_{v^i_\alpha} \tag{2}$$

With a lagrangian $L : N \to \mathbb{R}$ the authors proceed as before:

i. Variational calculus
The action to be considered is

$$\begin{array}{rccc} \mathcal{B}: & C^\infty_c(\mathbb{R}^k, Q) & \longrightarrow & \mathbb{R} \\ & \varphi & & \displaystyle\int_{\mathbb{R}^k} L \circ \varphi^1 \ d^k x \end{array}$$

where $d^k x$ is the $k-$volume form, and $\varphi^1 : \mathbb{R}^k \to N$ is the first prolongation of φ.The field equations then are derived from:

$$\delta\mathcal{B} = 0$$

ii. Geometrical settings
It is define a k family of $1-$forms:

$$\theta^\alpha_L = dL \circ J^\alpha$$

and

$$\omega^\alpha_L = -d\theta^\alpha_L$$

called the Poincaré - Cartan forms. Moreover it is defined a second order $k-$vector field as $(X_\alpha) \in \mathfrak{X}^k(N)$ such that

$$J^\alpha(X_\alpha) = \Delta_\alpha \quad \forall\alpha$$

where $\Delta = \sum \Delta_\alpha$. Equivalently

$$\left(T\tau^k\right)_{(q,v)} (X_\alpha) = (v_\alpha)_q$$

where $(v_\alpha)_q$ is the tangent vector to Q at $q \in Q$ given by the coordinates of the point $(q, v) \in N$ that carries α as subindices.
The field equations come from the flow of the second order $k-$vector field that verify:

$$i_{X_\alpha}\omega^\alpha_L = dE$$

where $E = \Delta(L) - L$.

In any case the field equations are written for

$$\begin{array}{cccc}\varphi: & \mathbb{R}^k & \longrightarrow & N \\ & x & & \left(\psi^i(x), \psi^i_\alpha(x)\right)\end{array}$$

and takes the form

$$\frac{\partial^2 L}{\partial q^j \partial v^i_\alpha}(\varphi(x))\ \frac{\partial \psi^j}{\partial x^\alpha}(x) + \frac{\partial^2 L}{\partial v^j_\beta \partial v^i_\alpha}(\varphi(x))\ \frac{\partial^2 \psi^j}{\partial x^\beta \partial x^\alpha}(x) = \frac{\partial L}{\partial q^i}(\varphi(x))$$

2. k cosymplectic formalism

   (a) Hamiltonian field equations

   To get account of the independent parameters, the authors introduce the manifold $\widetilde{M} = \mathbb{R}^k \times M \simeq J^1(Q, \mathbb{R}^k)$ with coordinates $(x^\alpha, q^i, p^\alpha_i)$. And $\widetilde{M}$ is endowed with the family of k 1−forms $\eta^\alpha = dx^\alpha$, and the Liouville forms $\theta^\alpha = p^\alpha_i\ dq^i$ which gives rise to $\omega^\alpha = -d\theta^\alpha$. They all satisfy that

$$\begin{cases} dim\left(\overset{k}{\cap} Ker\ \omega^\alpha\right) = k \\ \overset{k}{\cap} Ker\ \eta^\alpha\ \overset{k}{\cap}\ Ker\ \omega^\alpha = 0 \\ \eta^\alpha|_V = \omega^\alpha|_{V\times V} = 0 \quad \forall \alpha \\ \overset{k}{\wedge} \eta^\alpha \neq 0 \end{cases}$$

   In the opposite direction, this properties serve to define a $k-$cosymplectic structure on a manifold of dimension $nk + n + k$.

   This structure also carries the Reeb vector, a canonical $k-$vector field $(R_\alpha) \in \mathfrak{X}^k(\widetilde{M})$ which satisfies that

$$\begin{cases} i_{R_\beta} \eta^\alpha = \delta^\alpha_\beta \\ i_{R_\beta} \omega^\alpha = 0 \end{cases}$$

   Now, with a hamiltonian $H : \widetilde{M} \to \mathbb{R}$ the authors proceed as before

      i. Variational calculus

      The action is

$$\begin{array}{cccc}\widetilde{\mathcal{A}}: & \Gamma_c(\mathbb{R}^k, \widetilde{M}) & \longrightarrow & \mathbb{R} \\ & \varphi & & \displaystyle\int_{\mathbb{R}^k} \varphi^* \theta^\alpha \wedge d^{k-1}x^\alpha - (\varphi^* H)\, d^k x\end{array}$$

      where $d^k x^\alpha$ is the $k-$volume form and $d^{k-1}x^\alpha = dx^1 \wedge \ldots \wedge \widehat{dx^\alpha} \wedge \ldots \wedge dx^k$. $\Gamma_c(\mathbb{R}^k, \widetilde{M})$ are the sections of the fiber $\widetilde{M} \to \mathbb{R}^k$ with compact support. The field equations are derived from the variation

$$\delta \widetilde{\mathcal{A}} = 0$$

ii. Geometrical settings

The field equations come from the flow of the $k-$vector field $(X_\alpha) \in \mathfrak{X}^k(\widetilde{M})$ that verify:

$$\left\{ \begin{array}{l} \eta^\alpha(X_\beta) = \delta^\alpha_\beta \\ i_{X_\alpha}\omega^\alpha = dH - R_\alpha(H)\ \eta^\alpha \end{array} \right.$$

In any case the field equations are written for

$$\begin{array}{cccc} \varphi: & \mathbb{R}^k & \longrightarrow & \widetilde{M} \\ & x & & (x^\alpha, \psi^i, \psi^\alpha_i) \end{array}$$

and takes the form

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial q^i}(\varphi(x)) = -\displaystyle\sum_{\alpha=1}^{k} \frac{\partial \psi^\alpha_i}{\partial x^\alpha}(x) \\ \\ \frac{\partial H}{\partial p^\alpha_i}(\varphi(x)) = \frac{\partial \psi^i}{\partial x^\alpha}(x) \end{array} \right.$$

(b) Lagrangian field equations

The manifold to work with is $\widetilde{N} = \mathbb{R}^k \times N \simeq J^1(\mathbb{R}^k, Q)$ with coordinates $(x^\alpha, q^i, v^i_\alpha)$. This manifold has the family of k $1-$forms $\eta^\alpha = dx^\alpha$; a canonical family of k tensor fields $J^\alpha \in \mathfrak{X}^1_1(\widetilde{N})$ define as extensions of (1)

$$J^\alpha = \partial_{v^i_\alpha} \otimes dq^i$$

and the Liouville vector field $\Delta \in \mathfrak{X}(\widetilde{N})$ define also as an extension of (2) as:

$$\Delta = v^i_\alpha\ \partial_{v^i_\alpha}$$

With a lagrangian $L : \widetilde{N} \to \mathbb{R}$ the authors proceed as before:

i. Variational calculus

The action to be considered is

$$\begin{array}{cccc} \widetilde{\mathcal{B}}: & C^\infty_c(\mathbb{R}^k, Q) & \longrightarrow & \mathbb{R} \\ & \varphi & & \displaystyle\int_{\mathbb{R}^k} L \circ \varphi^1\ d^k x \end{array}$$

where $d^k x$ is the $k-$volume form, and $\varphi^1 : \mathbb{R}^k \to \widetilde{N}$ is the first prolongation of φ.The field equations then are derived from:

$$\delta\widetilde{\mathcal{B}} = 0$$

ii. Geometrical settings

It is define a k family of $1-$forms:

$$\theta^\alpha_L = dL \circ J^\alpha$$

and

$$\omega_L^\alpha = -d\theta_L^\alpha$$

called the Poincaré - Cartan forms. Moreover it is define a second order $k-$vector field as $(X_\alpha) \in \mathfrak{X}^k(\widetilde{N})$ such that

$$\begin{cases} \eta^\alpha(X_\beta) = \delta_\beta^\alpha \\ J^\alpha(X_\alpha) = \Delta_\alpha \quad \forall \alpha \end{cases}$$

where $\Delta = \sum \Delta_\alpha$. Equivalently every integral section of (X_α) is the first prolongation of a map $\phi : \mathbb{R}^k \to Q$.
The field equations come from the map $\phi : \mathbb{R}^k \to Q$ whose first prolongation is the flow of the second order $k-$vector field that verify:

$$\begin{cases} \eta^\alpha(X_\beta) = \delta_\beta^\alpha \\ i_{X_\alpha}\omega_L^\alpha = dE + \frac{\partial L}{\partial x^\alpha} dx^\alpha \end{cases}$$

where $E = \Delta(L) - L$.
In any case the field equations are written for

$$\begin{array}{cccc} \varphi : & \mathbb{R}^k & \longrightarrow & \widetilde{N} \\ & x & & \left(\psi^\alpha(x), \psi^i(x), \psi_\alpha^i(x)\right) \end{array}$$

and takes the form

$$\frac{\partial^2 L}{\partial x^\alpha \partial v_\alpha^i}(\varphi(x)) + \frac{\partial^2 L}{\partial q^j \partial v_\alpha^i}(\varphi(x)) \frac{\partial \psi^j}{\partial x^\alpha}(x) + \frac{\partial^2 L}{\partial v_\beta^j \partial v_\alpha^i}(\varphi(x)) \frac{\partial^2 \psi^j}{\partial x^\beta \partial x^\alpha}(x) = \frac{\partial L}{\partial q^i}(\varphi(x))$$

The book also includes the Legendre transformation and the Hamilton - Jacobi equation in both formalism. It has many examples as the massive scalar field or the Maxwell's equations. And finally, they offer a brief chapter on multisymplectic formalism and its relation to $k-$cosymplectic one.

This book shows the results of many spanish mathematicians as Manuel de León, Alberto Ibort, Muñoz-Lecanda, Narcis Román Roy, Cariñena, Xavier Gràcia, Echeverría-Enríquez, and Marco Castrillón among others. Of course I include the two coauthors Modesto Salgado and Silvia Vilariño. By the way, this last person work at the "Centro Universitario de la Defensa (CUD)". Let me explain that this is a brand new place with just few years of activities. It has been created by the Spanish Army to offer a graduate course to the military persons. Now, continuing with non spanish authors I cite to Giuseppe Marmo, Sardanashvily, Gotay etc And finally, I would highlight the book by Saunders "The Geometry of Jets Bundles" as a basic reference which also appears in the index.