On some theorems of
Reiter/Varopoulos/Saeki/Helson, and
sets of synthesis

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Consider the group algebra \((L^1(\mathbb{R}^n), +, \ast))\) of the group \((\mathbb{R}^n, +)\). For a closed subset \(E\) of \(\mathbb{R}^n\), let \(J(E)\) be the closure in \(L^1(\mathbb{R}^n)\) of the ideal 
\[ \{ f \in L^1(\mathbb{R}^n) : \text{Supp}(\hat{f}) \cap E = \emptyset \} \]
where \(\hat{f}\) is the Fourier transform of \(f\). These are the smallest and the largest closed ideals of \(L^1(\mathbb{R}^n)\) with hull \(E\). If \(J(E) = k(E)\), the set \(E\) is said to be a set of synthesis. In this talk I will present a series of new/recent results about the following old but good theorems. Below \(S = \{ x \in \mathbb{R}^n : ||x|| = 1 \}\) is the unit sphere of \(\mathbb{R}^n\).

**Theorem-1** (Reiter, Math. Ann. (135), 1958). Let \(n \geq 3\) and \(E\) be a closed subset of \(\mathbb{R}^n\). Then the set \(F = E \cup S\) is a set of synthesis iff \(S \subseteq E\) and \(E\) is a set of synthesis.

The fact that \(S\) is not a set of synthesis does not explain the reason why this theorem holds. Whence the question:

**Question-1.** What makes that Reiter’s theorem holds?

**Theorem-2.** (Varopoulos, Proc. Phil. Soc. Camb. (62), 1966). For \(n = 3\), the equality \(J(S) \perp \cap C_0(\mathbb{R}^n) = k(S) \perp \cap C_0(\mathbb{R}^n)\) holds.

It is rare that for a closed set \(F\) that fails to be a set of synthesis the equality \(J(F) \perp \cap C_0(\mathbb{R}^n) = k(F) \perp \cap C_0(\mathbb{R}^n)\) holds. Whence the question:

**Question-2.** What makes that this equality holds?

A closed \(H \subseteq \mathbb{R}^n\) is said to be an Helson set if the restriction homomorphism \(\phi : L^1(\mathbb{R}^n) \to C_0(H), \phi(f) = \hat{f}|H,\) is surjective.

**Theorem-3.** (Saeki, J. Math. Soc. Jap. (21), 1969). If \(H\) is an Helson set of synthesis then, for any set of synthesis \(E\), the union \(H \cup E\) is a set of synthesis.

Given that whether the union of two sets of synthesis is a set of synthesis or not is not known, it is natural to wonder:

**Question-3.** What makes that Saeki’s theorem holds?

In this talk I shall try to answer these and some other related questions.

A couple of the results are extracted from joint works with E. Kaniuth (Germany) and with A. To-Ming Lau (Canada).

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