Quadratic Irrationals. An Introduction to Classical Number Theory
Author: Franz Halter-Koch
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Reviewer: Capi Corrales Rodrigáñez, Department of Algebra, Mathematics, UCM, Madrid

On the occasion of his nomination as Doctor Honoris Causa by Universidad Complutense de Madrid in 2006, Jean Pierre Serre gave a conference at the Colloquium of the Mathematics Department entitled *How the number of solutions to a system of equations modulo p changes with p.* Professor Serre started his intervention explaining that the objective of his talk was to illustrate that one can learn (and teach) beautiful and deep mathematics without leaving the realm of quadratic irrationals, that is, complex numbers which are non-rational solutions to quadratic equations with rational coefficients. The book under review offers another illustration of this fact.

The transformation group induced by the action of the group of invertible matrices with integral coefficients on the upper half complex plane via $z \mapsto \frac{az+b}{cz+z}$, is known as the modular group. The study of the action of the modular group on the quadratic irrationals, is key to understand, and describe in a systematic way, the relation between integral binary quadratic forms, binary Diophantine equations and the theory of quadratic orders, central objects of study in classical number theory. Halter-Koch starts his text explaining this action and, in doing so, he sets in a contemporary frame and in a unified language, many of the questions addressed by 18th and 19th century number theorists such as Gauss, Legendre, Lagrange and Dirichlet. This makes it possible for him, later on in the book, to present the work of these leading mathematicians in the context of the mathematics of their time, as well as to trace the influence of their questions and answers in contemporary mathematics.

The digit expansion of an irrational real number produces rational approximations with arbitrary exactitude, and a systematic way to produce such good approximations with small denominators is provided by continued fractions. Furthermore, the continued fraction of a real quadratic irrational z is ultimately periodic, and its period carries arithmetical information about z. Chapter 2 introduces the work of Euler, Lagrange and Galois on periodicity of continued fractions, and it also provides a detailed study of the solutions to Pell's equation via continued fractions and other related concepts, which will be used later on (in Chapter 5) to investigate rings of units or ideal class groups in quadratic orders.

Another classical tool in the study of quadratic equations with rational coefficients, is quadratic reciprocity and the theory around it. Besides a proof of the quadratic reciprocity law using Gauss sums, tools such as Gauss and Jacobi sums or quadratic characters are presented in Chapter 3. These tools will be used to study Dirichlet series in Chapter 4, Gauss' genus theory of binary quadratic forms in Chapter 6 and bi-quadratic residues in Chapter 7.

Chapter 4 is dedicated to proving Dirichlet's theorem of primes in arithmetic progression. To this purpose, multiplicative functions and Dirichlet's L-series are introduced, in such a way that it also paves the way for the investigations in Chapter 8 involving quadratic orders, binary quadratic forms and the proof of the analytic class number formula.

The basic algebraic structures built with quadratic irrationals are studied in Chapter 5: lattices, orders, ideals and ideal classes in quadratic number fields. These structures are used in Chapter 6, together with the properties of continued fractions, quadratic residues and quadratic characters studied in previous chapters, to prove the main theorems of Gauss' theory of binary quadratic forms. Finally, cubic and bi-quadratic residues and reciprocity laws are studied in Chapter 7.

Having exposed thoroughly along the first seven chapters the equivalence between ideal class groups of quadratic orders, equivalence classes of quadratic irrationals, and composition class groups of binary forms, the book closes with a final chapter dedicated to class groups. First, taking an analytic point of view, a proof of Dirichlet's analytic class number formula is given. Next, following an algebraic approach, the 2-component of the class group of a quadratic order is studied.

The number theory of quadratic irrationals is a classical object of study, and most of the basic results presented in Franz Halter-Koch's book can be found in many textbooks on elementary number theory. However, the material is usually scattered through the different chapters, the notation is not standardized and no unifying approach for the different aspects of the subject is taken. That is precisely the value of the volume under review, and what makes it specially interesting as a text book on elementary number theory: the material is organized in a consisted way and the language is unified, making it possible for students and readers to learn about one of the classical theories developed in number theory, and do it in such a way that not only the reason for and relation between the different points of view followed through time is understood, but so are some important and beautiful recent results and approaches. In addition, the necessary algebraic and analytical background is introduced in two final appendices, making the text accessible for readers with modest backgrounds in elementary algebra and number theory.