A primer on mapping class groups

Given a compact connected orientable surface S there are two fundamental objects attached: a group and a space.

The group is the *mapping class group* of S, denoted by $Mod(S)$. This group is defined by the isotopy classes of orientation-preserving homeomorphism from S to itself. Equivalently, $Mod(S)$ may be defined using diffeomorphisms instead of homeomorphisms or homotopy classes instead of isotopy classes.

The space is the *Teichmüller space* of S, $Teich(S)$. Teichmüller space and moduli space are fundamental objects in fields like low-dimensional topology, algebraic geometry and mathematical physics. If $\mathcal{X}(S) < 0$, the Teichmüller space can be thought of as the set of homotopy classes of hyperbolic structures of S or, equivalently, as the set of isotopy classes of hyperbolic metrics on S, $\mathrm{HypMet}(S)$.

The group and the space are connected through the *moduli space* in the following way. The group of orientation-preserving diffeomorphisms of S, $\mathrm{Diff}^+(S)$ acts on $\mathrm{HypMet}(S)$ and this action descends to an action of $Mod(S)$ on $\mathrm{Teich}(S)$ which is properly discontinuous. The quotient space,

$$\mathcal{M}(S) = \mathrm{Teich}(S)/Mod(S)$$

is the moduli space of Riemann surfaces homeomorphic to S.

One of the main results proved in the book is the Nielsen-Thurston classification theorem. This theorem establishes that each element of the mapping class group has a representative which is either periodic, reducible and pseudo-Anosov. In some sense, this result is analogous to the Jordan canonical form for matrices.

The proof presented here follows Bers' proof. It uses many ingredients of the interplay between mapping class groups and Teichmüller space which are developed in the book.

The book under review has three parts.

Part I is dedicated to the algebraic structure of the mapping class groups. It begins with a review of the basics on surfaces, simple closed curves and hyperbolic geometry.

Then, the authors define the mapping class group and compute it on the examples where this can be explicitly done: some punctured spheres, annulus, torus and one-punctured torus. They also introduce the *Alexander method*, an algorithm to determine whether or not two elements of $Mod(S)$ are equal.

Next, the authors discuss the algebraic structure and properties of the group. In particular, it is proved that $Mod(S)$ is finitely generated by infinite order elements of $Mod(S)$ called *Dehn twists* which are easily visualized. They also analyze the torsion and presentations of these groups.

They also analyze the *Dehn-Nielsen-Baer theorem*, which gives an isomorphism between a topologically defined group, the extended mapping class group of a surface (including reversing orientation homeomorphisms), and an algebraically defined group, the group of outer automorphisms of the surface's fundamental group.

This part finishes with a brief introduction to *braid groups*.

Part II is an introduction to Teichmüller theory and the moduli space, focusing in the aspects more connected to the mapping class group. Considering Teichmüller space as the homotopy classes of hyperbolic structures on S, they introduce Fenchel-Nielsen coordinates and prove that Teichmüller space is topologically a ball. Next they introduce the Teichmüller metric, making use of the theory of quasi-conformal maps and quadratic differentials, and describe the Teichmüller geodesics. Finally, they explain the proper discontinuous action of the mapping class group in Teichmüller space by isometries, giving as quotient the moduli space. The topology at infinity of the moduli space and Mumford's compactness criterion will be important for the Nielsen-Thurston classification theorem, so they are analyzed. In addition, they comment another important aspect of moduli space: moduli space is very close to be a classifying space for surface bundles.

Part III is devoted to the Nielsen-Thurston classification theorem. There is a nice explanation of the history of this theorem and about the motivation of this classification result for 3-dimensional topology. The proof presented with details in the book follows the one given by Bers. Nevertheless, they present the many ideas of Thurston's proof in a different chapter.

The most important type of surface homeomorphisms are the pseudo-Anosov ones. There is a whole chapter dedicated to their construction (in several different ways) and their properties.

We would like to include a few comments about the book.

The notation for the mapping class group is not standard in the literature. Here the authors use the original notation, $Mod(S)$, coming from the terminology "modular group" since the mapping class group is a generalization of the classical modular group $SL(2, \mathbb{Z})$. Since moduli space, denoted in the text $\mathcal{M}(S)$, is a main topic in the book, this notation might sometimes result a bit confusing. Other usual notations in the literature for the mapping class group are $\mathcal{M}(S)$, $MCG(S)$, $Map(S)$ and Γ.

The book is written with a clear intention to make this matter accessible for a wide audience. It is worth mentioning that the book begins with an *Overview* chapter summarizing, chapter by chapter, the contents of the book.

The text is also very didactic. It presents the most important theorems on the subject in such a way that the reader can easily follow the ideas behind the theory. For instance:

– they give intuitive explanations before the rigorous proofs;

– they give explanations about why some things work and others don't;

– they explain why they choose some definitions instead others;

– they give different points of view of the same concept;

– there are plenty of indications of the use of some results as ingredients of other bigger results.

The explanations presented are direct and clear and they are kept as simple as possible, without falling in big technicalities. Of course this may imply that some proofs are not perfectly finished including every detail, but it is usually clear how to fill the gaps.

Also, there are interesting remarks about the history of the theory presented and about the interconnections of some results with others.

In summary, this is a very pleasant and appealing book and it is an excellent reference for any reader willing to learn about this fascinating part of mathematics.