ABSTRACT. Let \( N \) be a positive integer and \( a_1, \ldots, a_N \) be complex numbers. We denote by \( S(N) \) the supremum of the ratio between \( |a_1| + \cdots + |a_N| \) and \( \sup_{t \in \mathbb{R}} |a_1 + a_2 e^{-it} + \cdots + a_N e^{-iNt}| \), with the supremum taken over all possible choices of nonzero vectors \((a_1, \ldots, a_N)\) in \( \mathbb{C}^N \). In a series of four essentially independent lectures, I will discuss the following remarkable result:

\[
S(N) = \sqrt{N} \exp \left( (-\frac{1}{\sqrt{2}} + o(1)) \sqrt{\log N \log \log N} \right)
\]

when \( N \to \infty \). This formula has a long history and relies on the contribution of many researchers, including H. Bohr, Bohnenblust–Hille, Queffélec, Queffélec–Konyagin, de la Bretèche, and finally Defant–Frerick–Ortega-Cerdà–Ounaïes–Seip. The proof involves several interesting techniques that will be highlighted and discussed during the lectures.

In the paper [HLS97] we studied Dirichlet series in \( \mathcal{H}^2 \). This space consists of the series

\[
f(s) = \sum_{n=1}^{\infty} a_n n^{-s}
\]

with

\[
\|f\|_2^2 = \sum_{n=1}^{\infty} |a_n|^2 < +\infty.
\]

Letting \( s = \sigma + it \), by the Cauchy-Schwarz inequality,

\[
|f(s)|^2 \leq \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} n^{-2\sigma},
\]

so \( f(s) \) defines an analytic function in the right half-plane \( \{ s \in \mathbb{C} : \sigma > 1/2 \} \).

An easy computation shows that the reproducing kernel for the space \( \mathcal{H}^2 \) is \( k_w(s) = \zeta(\bar{w} + s) \), where \( \zeta(s) \) is the Riemann zeta function.

A main result in [HLS97] is that the multipliers of \( \mathcal{H}^2 \) are

\[
\mathcal{H}^\infty := H^\infty(\mathbb{C}_+) \cap \mathcal{H}^2,
\]

where \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re } z > 0 \} \).

There is an interesting difference in convergence between Dirichlet series in \( \mathcal{H}^2 \) and \( \mathcal{H}^\infty \): the series defining a function \( f \in \mathcal{H}^\infty \) is absolutely convergent in \( \{ \sigma = 1/2 \} \). In fact, for every...
As shown in [BCQ06], this is equivalent to

\[(1) \quad S(N) := \sup_{a = (a_1, \ldots, a_N) \neq 0} \left\| \sum_{n=1}^{N} a_n n^{-s} \right\|_{\infty} = \sqrt{N} \exp \left( - \frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log N \log \log 2N} \]

The sup-norm here is taken on \( \sigma = 0 \) and \( \log_2 N = \log \log N \).

The value \( S(N) \) is known as the Sidon constant (of the set \( \log 1, \log 2, \ldots, \log N \)). Konyagin and Queffélec [KQ01/02] and de la Bretèche [dlB08] gave estimates on this value, and Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip [DFOCOS11] obtained the final upper estimate.

Using that

\[\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{i \log m t} dt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \]

one sees that for Dirichlet polynomials

\[\sum_{n=1}^{N} |a_n| \leq \sqrt{N} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2} = \sqrt{N} \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \sum_{n=1}^{N} a_n n^{-it} \right|^2 dt \right)^{1/2} \leq \sqrt{N} \left\| \sum_{n=1}^{N} a_n n^{-s} \right\|_{\infty} .\]

Thus, the extra factor in (1) can be viewed as a quantitative statement of how “flat” Dirichlet polynomials can be.

Let us compare this with the corresponding situation for trigonometric polynomials \( Q(z) = \sum_{n=0}^{N} a_n z^n \). Kahane showed in [K80] that these can be “ultra-flat”, in the sense that it is possible to choose \( a_n \) with \( |a_n| = 1 \) and

\[|Q(e^{it})| = \sqrt{N} (1 + o(1)), \quad N \to \infty .\]

Bombieri and Bourgain refined this statement by showing that \( o(1) \) is actually of the form \( N^{-1/9+\epsilon} \) [BB09].

**Bohr’s correspondence.**

Consider the prime numbers \( p_1, p_2, \ldots \), written in increasing order. For each \( n \in \mathbb{N} \) denote by \( \alpha(n) = (\nu_1(n), \nu_2(n), \ldots) \) the multi-index for which \( n = p_1^{\nu_1(n)} p_2^{\nu_2(n)} \cdots \). Notice that in \( \alpha(n) \) there are only a finite number of non-zero entries. Then

\[n^{-s} = (p_1^{-s})^{\nu_1(n)} (p_2^{-s})^{\nu_2(n)} \cdots = z_1^{\nu_1(n)} z_2^{\nu_2(n)} \cdots = z^{\alpha(n)} ,\]
where \( z = (z_1, \ldots, z_n) = (p_1^{-s}, p_2^{-s}, \ldots) \).

By this correspondence a Dirichlet polynomial \( f(s) \) of degree \( N \) is identified with an algebraic polynomial \( F(z) \) in at most \( \pi(N) \) variables, and

\[
\sum_{n=1}^{N} |a_n|^2 = \int_{\mathbb{T}^\pi(N)} |F(z)|^2 d\mu(z),
\]

where \( d\mu \) denotes the normalised Lebesgue measure. Then

\[
\|F\|_\infty = \lim_{k \to \infty} \left( \int_{\mathbb{T}^\pi(N)} |F(z)|^{2k} d\mu(z) \right)^{\frac{1}{2k}} = \lim_{k \to \infty} \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T |f(it)|^{2k} dt \right)^{\frac{1}{2k}} = \|f\|_\infty.
\]

The last identity follows from the almost-periodicity of \( f \). Actually, the image of the imaginary axis under Bohr’s correspondence is dense and equidistributed (with respect to the Lebesgue measure) in \( \mathbb{T}^\infty \) or in \( \mathbb{T}^\pi(N) \).

**Lower estimate of Sidon’s constant through Rudin–Shapiro polynomials.**

Maurizi and Queffélec showed how to obtain a lower estimate of \( S(N) \) using the Rudin–Shapiro homogeneous polynomials [MQ10].

Consider the Hadamard matrix

\[
A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

which satisfies \( A_1^2 = A_1^* A_1 = 2I \). Define \( P_0 = Q_0 = 1 \) and start the recursion

\[
(P_{j+1}, Q_{j+1}) = A_1 (P_j, z^{2^j} Q_j).
\]

Thus

\[
P_1 = 1 + z, \quad Q_1 = 1 - z,
\]

\[
P_2 = 1 + z + z^2 - z^3, \quad Q_2 = 1 + z - z^2 + z^3.
\]

In general \( P_j, Q_j \) are polynomials of degree \( 2^j - 1 \) whose \( 2^j \) coefficients are \( \pm 1 \), and

\[
|P_j(z)|^2 + |Q_j(z)|^2 = 2^{j+1}.
\]

Since \( \|P_j\|^2_2 = 2^j \) we have

\[
\|P_j\|_\infty \leq \sup_{z \in \mathbb{T}} (|P_j(z)|^2 + |Q_j(z)|^2) = 2 \|P_j\|^2_2,
\]

so

\[
\|P_j\|_\infty \leq \sqrt{2} \|P_j\|_2.
\]

An analogous construction can be done with Hadamard matrices of higher order. Let \( A_k \) be a \( 2^k \times 2^k \) Hadamard matrix with \( A_k^2 = A_k^* A_k = 2^k I \) and define

\[
A_{k+1} = \begin{pmatrix} A_k & A_k \\ A_k & -A_k \end{pmatrix}.
\]
Let $q = 2^k$ and start with 
\[
(P_{0,1}, P_{0,2}, \ldots, P_{0,q}) = (1, 1, \ldots, 1).
\]

Define recursively 
\[
(P_{j+1,1}, P_{j+1,2}, \ldots, P_{j+1,q}) = A_k(z_{jq+1}P_{j,1}, \ldots, z_{jq+q}P_{j,q}).
\]

Notice that at each step $q$ new variables are introduced.

The basic properties of these polynomials are:

- $P_{m,l}$ is a homogeneous polynomial of degree $m$ in $n = mq$ variables whose $q^m$ coefficients are $\pm 1$,
- $|P_{m,1}(z)|^2 + \cdots + |P_{m,q}(z)|^2 = q^{m+1}$.

Here the upper bound of $\|P_{m,1}\|_2/\|P_{m,1}\|_\infty$ is $\sqrt{q}$ instead of $\sqrt{2}$:
\[
\|P_{m,1}\|_\infty \leq q^{m+1} = q^{1/2}\|P_{m,1}\|_2 \quad (= q^{-m+1}. \# \text{ of coefficients}).
\]

We want to use Bohr’s correspondence and the Rudin–Shapiro polynomials $P_{m,l}$ to produce a Dirichlet polynomial of degree at most $N$, which will be used to estimate $S(N)$. The number of variables $n = mq$ corresponds to the number of primes appearing in the Dirichlet polynomial. Also, since $P_{m,l}$ are homogeneous of degree $m$, we can only use prime numbers $p_i$ with $p_i^m \leq N$.

Consider thus $x$ such that $x^m = N$ and look for the largest homogeneous polynomial in $n$ variables made with primes in $(1, x)$. By the prime number theorem
\[
n = mq = \pi(x) + o(1) = \frac{x}{\log x} + o(1) = \frac{N^{1/m}}{m \log N} + o(1),
\]
and therefore
\[
q = \frac{N^{1/m}}{\log N} (1 + o(1)).
\]

This provides a lower estimate of the Sidon constant:
\[
S(N) \geq q^{-m+1} = (1 + o(1)) N^{1/2} \frac{N^{-\frac{1}{2m}}}{(\log N)^{\frac{m-1}{2}}}.
\]

A computation (calculus) shows that the factor
\[
\frac{N^{-\frac{1}{2m}}}{(\log N)^{\frac{m-1}{2}}} = \exp \left\{ -\frac{1}{2} \left( \frac{\log N}{m} + m \log N \right) \right\}
\]
is maximal when $m = \sqrt{\frac{\log N}{\log_2 N}}$. Therefore
\[
S(N) \geq \exp \left\{ -\sqrt{\log N \log_2 N} \right\}.
\]

Thus, with Hadamard matrices we get close to the sharp estimate given in (1) (instead of the value $-1/\sqrt{2}$ we get $-1$). There is a loss in the restriction imposed by considering only the primes between 1 and $x$. In order to get the sharp constant more number theory is necessary.
Homogeneous polynomials. The Bohnenblust–Hille inequality.

Rudin’s book on function theory in the polydisc [R69] aims to extend function theory results in $\mathbb{D}$ to the polydisc. This may explain why the Bohnenblust–Hille inequality is not mentioned.

Let $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ be a homogeneous polynomial of degree $m$ in $\mathbb{D}^n$. By integration on the distinguished boundary $\mathbb{T}^n$ we see that

$$(\sum_{|\alpha|=m} |a_\alpha|^2)^{1/2} \leq \|P\|_\infty = \sup_{z \in \mathbb{D}^n} |P(z)| .$$

**Question.** Is it possible to have $C = C(m, p) > 0$, independent of $n$, such that

$$(\sum_{|\alpha|=m} |a_\alpha|^p)^{1/p} \leq C\|P\|_\infty$$

for some $p < 1/2$?

The answer is affirmative, and the smallest value of $p$ for which such a constant exists is $p = \frac{2m}{m+1}$ (Bohnenblust–Hille [BH31]).

The natural question now is: what is the best possible $C$ for this choice of $p$?

The Rudin–Shapiro polynomials $P$ already indicate that $p = \frac{2m}{m+1}$ is the best possible choice: since $\|P\|_\infty \leq q^{m+1/2}$ and

$$(\sum_{|\alpha|=m} |a_\alpha|^p)^{1/p} = q^{m/p}$$

the best choice follows from taking $p$ with $\frac{m}{p} = \frac{m+1}{2}$.

Bohr’s correspondence relates $i\mathbb{C}^n$ with $\mathbb{D}^n$, and the Bohnenblust–Hille inequality relates $\mathbb{D}^n$ with $\mathbb{D}^n \times \cdots \times \mathbb{D}^n$.

**Littlewood’s estimate** [Li30]. Let $\{e^{(i)}\}_{i=1}^n$ denote the canonical basis of $\mathbb{C}^n$. If $B : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ is a bilinear form then

$$\left( \sum_{i,j} |B(e^{(i)}, e^{(j)})|^{4/3} \right)^{3/4} \leq \sqrt{2} \sup_{z,w \in \mathbb{D}^n} |B(z, w)| = \sqrt{2}\|B\|_\infty .$$

This extends to multi-linear forms.

**Bohnenblust–Hille estimate:** If $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \rightarrow \mathbb{C}$ is an $m$-linear form then

$$\left( \sum_{i_1, \ldots, i_m} |B(e^{(i_1)}, \ldots, e^{(i_m)})|^{2m/(m+1)} \right)^{m+1/2m} \leq (\sqrt{2})^{m-1}\|B\|_\infty ,$$

and this is the best possible.

There is a 1-1 correspondence between homogeneous polynomials of degree $m$ and $m$-linear symmetric forms. An $m$-linear form $B$ gives a homogeneous polynomial of degree $m$ just by

$P(z) = B(z, \ldots, z) . $
To go in the other direction we use polarisation. This can be done in two ways.

1st way (Bohnenblust–Hille’s way). Define

\[ B(z^{(1)}, \ldots, z^{(m)}) = \frac{1}{m!} \int_{\mathbb{T}^m} P(\zeta_1 z^{(1)} + \cdots + \zeta_m z^{(m)}) \, \bar{\zeta}_1 \cdots \bar{\zeta}_m \, d\mu(\zeta), \]

where \( d\mu \) denotes the normalised Lebesgue measure.

This is clearly symmetric, and by the homogeneity of \( P \) and the orthogonality of the products \( \zeta^\alpha \bar{\zeta}^\beta \),

\[ B(z, \ldots, z) = \frac{1}{m!} P(z) \int_{\mathbb{T}^m} (\zeta_1 + \cdots + \zeta_m)^m \bar{\zeta}_1 \cdots \bar{\zeta}_m \, d\mu(\zeta) = \frac{1}{m!} P(z) \int_{\mathbb{T}^m} m! \zeta_1 \cdots \zeta_m \bar{\zeta}_1 \cdots \bar{\zeta}_m \, d\mu(\zeta) = P(z). \]

Since \( \zeta_1 z^{(1)} + \cdots + \zeta_m z^{(m)} \in m\mathbb{D}^n \), from the definition above and the homogeneity we obtain the following estimate.

Polarisation lemma:

\[ \|B\|_\infty \leq \frac{m^m}{m!} \|P\|_\infty \leq e^m \|P\|_\infty. \]

2nd way. Let \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and assume the polynomial is given in the following form

\[ P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha = \sum_{i_1 \leq \cdots \leq i_m} c_{i_1, \ldots, i_m} z_{i_1} \cdots z_{i_m}, \quad i_j \in \{1, \ldots, n\}. \]

Given any \( i = (i_1, \ldots, i_m) \) (not necessarily ordered) denote by \( i_0 \) the permutation of these indices for which the entries are arranged in increasing order. In this notation \( P(z) = \sum_{i_0} c_{i_0} z_{i_0} \). In order to have an \( m \)-linear form we should have the same coefficients whenever the indices in \( i_0 \) are permuted. Denote by \( |i| \) the number of permutations of these indices; then the form must have coefficients \( B(e^{(i_1)}, \ldots, e^{(i_m)}) = \tilde{c}_1 = c_{i_0}/|i|. \) Since there might be repetitions, we have \( |i| \leq m! \), with possibility of strict inequality.

Estimating the form coefficients by the polynomial we have

\[
\left[ \sum_i |\tilde{c}_i|^{\frac{2m}{m+1}} \right]^{m+1 \over 2m} = \left[ \sum_i \left( \frac{|c_{i_0}|}{|i|} \right)^{2m} \right] \left( \sum_{i_0} |c_{i_0}| \right)^{2m} \left( \sum_{i_0} \left| c_{i_0} \right|^{2m} \right) \geq (m!)^{-m+1 \over 2m} \left( \sum_{i_0} |c_{i_0}|^{2m} \right)^{m+1 \over 2m}.
\]

The Bohnenblust–Hille inequality for forms gives now

\[
\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{m+1 \over 2m} = \left( \sum_{i_0} |c_{i_0}|^{\frac{2m}{m+1}} \right)^{m+1 \over 2m} \leq (\sqrt{2})^{m-1} (m!)^{m+1 \over 2m} \|B\|_\infty.
\]
With this and the polarisation lemma we finally get
\[
\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq (\sqrt{2})^{m-1} e^m (m!)^{1/2} \|P\|_\infty .
\]

We want a polynomial inequality of type
\[
(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C^m \|P\|_\infty ,
\]
so we still have an extra factor \((m!)^{1/2}\).

**Proof of the Bohnenblust–Hille inequality for forms.** Pick \(k \in \{1, \ldots, m\}\) and write \(B\) as a combination of \((m-1)\)-linear forms:
\[
B(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_k=1}^n z^{(k)}_i \cdot B^{(k)}_i(z^{(1)}, \ldots, \widehat{z^{(k)}}, \ldots, z^{(m)})
\]
where \(\widehat{z^{(k)}}\) means that the variable \(z^{(k)}\) does not appear.

We have
\[
\|B\|_\infty = \sup_{z^{(1)}, \ldots, z^{(m)}} \sum_{i_k=1}^n |B^{(k)}_i| \geq \sum_{i_k=1}^n \|B^{(k)}_i\|_{L^1(T^{m-1})} .
\]
Here we use the following Khinchine inequality [Sa85]: for an \(m\)-linear form \(B\)
\[
\|B\|_2 \leq (\sqrt{2})^m \|B\|_1 .
\]
Thus
\[
\|B\|_\infty \geq (\sqrt{2})^{1-m} \sum_{i_k=1}^n \|B^{(k)}_i\|_2 .
\]
Combining these inequalities for all \(k\) (taking the geometric mean) we obtain
\[
\|B\|_\infty \geq (\sqrt{2})^{1-m} m \left( \sum_{i_k=1}^n \|B^{(k)}_i\|_2 \right)^{1/m} ,
\]
which is
\[
\prod_{k=1}^m \left[ \sum_{i_k=1}^n \left( \sum_{i_1, \ldots, i_m} |\tilde{c}_1|^2 \right)^{1/2} \right]^{1/m} \leq (\sqrt{2})^{m-1} \|B\|_\infty .
\]

The final step is the application of **Blei’s inequality** [Bl79]
\[
\left( \sum_{1}^n |\tilde{c}_1|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \prod_{k=1}^m \left[ \sum_{i_k=1}^n \left( \sum_{i_1, \ldots, i_m} |\tilde{c}_1|^2 \right)^{1/2} \right]^{1/m} ,
\]
which yields

\[
\left( \sum_i |\tilde{c}_i|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq (\sqrt{2})^{m-1} \|B\|_\infty,
\]

as desired. \(\square\)

**Proof of Blei’s inequality.** This is done by repeated use of Hölder and Minkowski’s inequalities. Let us see it for \(m = 2\). Using first Hölder’s inequality with exponents \(3/2\) and \(3\) on the sum on \(j\) we have

\[
\sum_i \sum_j |\tilde{c}_{i,j}|^{4/3} = \sum_i \sum_j |\tilde{c}_{i,j}|^{2/3} |\tilde{c}_{i,j}|^{2/3} \leq \sum_i \left( \sum_j |\tilde{c}_{i,j}| \right)^{2/3} \left( \sum_j |\tilde{c}_{i,j}|^2 \right)^{1/3}.
\]

Applying again Hölder’s inequality, now on the sum on \(i\) with exponents \(3\) and \(3/2\), and then Minkowski’s inequality on the first term we get

\[
\sum_i \sum_j |\tilde{c}_{i,j}|^{4/3} \leq \left[ \left( \sum_i \left( \sum_j |\tilde{c}_{i,j}| \right)^2 \right)^{1/3} \right] \left[ \left( \sum_i \left( \sum_j |\tilde{c}_{i,j}|^2 \right)^{1/2} \right)^{2/3} \right].
\]

\[
\leq \left[ \left( \sum_j \left( \sum_i |\tilde{c}_{i,j}|^2 \right)^{1/2} \right)^{2/3} \right] \left[ \left( \sum_i \left( \sum_j |\tilde{c}_{i,j}|^2 \right)^{1/2} \right)^{1/2} \right]^{2/3}.
\]

\(\square\)

**Remarks.** 1. To prove the polynomial inequality (2) the same elements are used: reduction, Khinchine, Blei, plus the polarisation lemma.

2. There is a new proof, based on methods of operator theory, for the estimate for multi-linear forms \([DPS10]\).

3. What about the constants? Can the factor \((\sqrt{2})^{m-1}\) be improved? There are no lower bounds known, other than an absolute constant.

Let’s go back to the description of the asymptotic growth of \(S(N)\) given in (1).

**Proof of \(\leq\).** (Konyagin–Queffélec \([KQ01/02]\))

The idea here is to write the Dirichlet polynomial as a sum of homogeneous polynomials (in the polydisc, after Bohr’s correspondence). If one writes

\[
\sum_{n=1}^{N} a_n n^{-s} = \sum_m \sum_{\Omega(n) = m} a_n n^{-s},
\]

where \(\Omega(n)\) denotes the number of prime factors in \(n\) (counted according to their multiplicities), it turns out that there are somehow too many terms.

We try to restrict to large primes (in contrast to the previous restriction to small primes). We will choose \(y\) between 1 and \(N\) and restrict to primes larger than \(y\). Write \(n = kl\) in such a way
that
\[ p \mid k \quad \Rightarrow \quad p \leq y \quad \text{(small)} \]
\[ p \mid l \quad \Rightarrow \quad p > y \quad \text{(large)}. \]

Now write
\[ f(s) = \sum_{n=1}^{N} a_n n^{-s} = \sum_{k} \sum_{l} a_{kl} k^{-s} l^{-s} := \sum_{k} f_k(s), \]
where
\[ f_k(s) = (\sum_{l} a_{kl} l^{-s}) k^{-s}. \]

Writing \( f_k \) as the integral of \( f \) against the corresponding monomial we see that
\[ \|f_k\|_{\infty} \leq \|f\|_{\infty}. \]

Let us see that, with an appropriate choice of \( y \), it is enough to prove the estimate for the \( f_k \)'s. Assume we have, for all \( f_k(s) \), the estimate
\[ \sum_{l} |a_{kl}| \leq \|f_k\| \exp \left( \left( -\frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log N \log \log N} \right). \]

Then
\[ \sum_{n} |a_n| = \sum_{k} \sum_{l} |a_{nk}| \leq \sum_{k} \|f_k\|_{\infty} \exp \left( \left( -\frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log N \log \log N} \right) \]
\[ \leq \|f\|_{\infty} \exp \left( \left( -\frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log N \log \log N} \right) \#\{k : f = \sum_{k} f_k\}, \]
and we will get the desired estimate as soon as we choose \( y \) small enough so that the number of \( N \) with prime factors only in \((1, y)\) is of size \( \exp \{ o(1) \sqrt{\log N \log \log N} \} \). A rough estimate will be enough.

Since \( 2 \) is the smallest prime, the number of prime factors of any number smaller than \( N \) is clearly bounded by the biggest \( j \) such that \( 2^j \leq N \), i.e. \( j \leq \frac{\log N}{\log 2} \). We have thus \( \frac{\log N}{\log 2} \) slots to be filled with \( y \) numbers, which gives
\[ \left( \frac{\log N}{\log 2} \right)^y = \exp \{ y(\log_2 N - \log_2 2) \} \]
possibilities. It is fine to choose
\[ y = (\log N)^{\delta} \quad \text{with} \quad \delta < 1/2. \]

From now on we assume that \( f \) is one of the \( f_k \), that is, that all \( n \) in \( f \) are \( y \)-large (meaning that all prime factors of \( n \) are bigger than \( y \)).

Let
\[ E_m = \{ n \leq N : \Omega(n) = m, \ n \text{ is } y \text{-large, } n \in \mathbb{N} \}. \]
Write, for a given \( M \) to be chosen later,

\[
(3) \quad f(s) = \sum_{m=1}^{M} \sum_{n \leq N \mid \Omega(n) = m} a_n n^{-s} + \sum_{\Omega(n) > M \atop n \leq N} a_n n^{-s}
\]

Let us estimate these two terms separately. For the latter sum, by Cauchy-Schwarz,

\[
\sum_{\Omega(n) > M \atop n \leq N} |a_n| \leq \left( \sum_n |a_n|^2 \right)^{1/2} \leq \|f\|_\infty \left( \sum_{m > M} |E_m| \right)^{1/2}.
\]

Here we will use the Bohnenblust–Hille inequality and some precise estimates of \(|E_m|\). These can be obtained by the so-called Rankin’s trick. Given \( c > 0 \) write

\[
\sum_{m \geq M} |E_m| \leq \frac{1}{(cy)^M} \sum_{\Omega(n) \geq M \atop n \leq N} (cy)^\Omega(n) \leq \frac{N}{(cy)^M} \sum_{\Omega(n) \geq M \atop n \leq N} \frac{(cy)^\Omega(n)}{n} \leq \frac{N}{(cy)^M} \prod_{y < p \leq N} \left( 1 - \frac{cy}{p} \right)^{-1}.
\]

Taking \( c = 1/2 \) is ok. Since \((1 - x)^{-1} \leq e^{2x} \) for \( x \in (0, 1/2) \), now we can estimate

\[
\prod_{y < p \leq N} \left( 1 - \frac{cy}{p} \right)^{-1} \leq \exp \left( 2cy \sum_{y < p \leq N} \frac{1}{p} \right) \leq \exp \left\{ 2c(\log N)^\delta (\log_2 N + O(1)) \right\}.
\]

Since \( \delta < 1/2 \) all this can go to the “\( o(1) \)” terms.

Also, for the factor \( N/(cy)^M \) we have

\[
\left( \frac{N}{(cy)^M} \right)^{1/2} = \sqrt{N} \exp \left\{ - \frac{M}{2} (\log c + \log y) \right\} = \sqrt{N} \exp \left\{ - \frac{M}{2} (1 + o(1)) \delta \log_2 N \right\}
\]

Choosing \( M = \frac{\sqrt{2}}{\delta} \sqrt{\frac{\log N}{\log_2 N}} \) we get

\[
\left( \frac{N}{(cy)^M} \right)^{1/2} = \sqrt{N} \exp \left\{ - \frac{1}{\sqrt{2}} (1 + o(1)) \sqrt{\log N \log_2 N} \right\},
\]

which yields the right estimate for the second term in (3):

\[
\frac{\sum_{\Omega(n) > M \atop n \leq N} |a_n|}{\|f\|_\infty} \leq \left( \sum_{m \geq M} |E_m| \right)^{1/2} \leq \left[ \frac{N}{(cy)^M} \right]^{1/2} \left[ \prod_{y < p \leq N} \left( 1 - \frac{cy}{p} \right)^{-1} \right]^{1/2} \leq \sqrt{N} \exp \left\{ - \frac{1}{\sqrt{2}} (1 + o(1)) \sqrt{\log N \log_2 N} \right\} \exp \left\{ 2c(\log N)^\delta (\log_2 N + O(1)) \right\}.
\]
On the other hand, to estimate the first term in (3) we apply first Hölder’s inequality
\[ \sum_{\Omega(n)=m, n \leq N} |a_n| \leq \left( \sum_n |a_n|^{2m/(2m+1)} \right)^{2m/(2m+1)} |E_m|^{m/(2m+1)}. \]

From this and the Bohnenblust–Hille inequality we have
\[ \sum_{\Omega(n)=m, n \leq N} |a_n| \leq C^m \|f\|_\infty |E_m|^{m/(2m+1)}. \]

The factor $C^m$ also goes to the $o(1)$ term:
\[ C^m \leq \exp(M \log C) = \exp \left( o(1) \sqrt{\log N \log \log N} \right). \]

The estimate of $|E_m|^{m/(2m+1)}$ for small $m$ is easy. Since $|E_m|^{m/(2m+1)} \leq N^{1/2} N^{-1/(2m)}$ and
\[ N^{-1/(2m)} = \exp \left( -\frac{1}{2m} \log N \right) \]
we have the right estimate as soon as $m \leq \frac{1}{\sqrt{2}} \sqrt{\log N/\log_2 N}$.

It remains to take care of the $m$ between this value and $M$, that is,
\[ m \in \left[ \frac{1}{\sqrt{2}} \sqrt{\log N/\log_2 N}, \frac{\sqrt{2}}{\delta} \sqrt{\log N/\log_2 N} \right]. \]

This is done with the more precise estimate given by Rankin’s trick ($|E_m| \lesssim N^{1/(2m)}$), which gives the desired result. (Here we use that we may choose $\delta = 1/2 + o(1)$ when $N \to \infty$, and we use calculus to find that the “worst” $m$ is $\sqrt{2} \sqrt{\log N/\log_2 N}$.)

**What about $\geq$?**

Let us just indicate how this is done, without entering into the details.

Previously we considered $x$ between 1 and $N$ and looked at the largest homogeneous polynomials made with primes in $(1, x)$. Crude estimates with Rudin–Shapiro polynomials gave the right estimate, except for the value 1 instead of $1/\sqrt{2}$ (see (1)). It is alright to restrict to small primes, but it’s too crude to consider only homogeneous polynomials.

De la Bretèche used a probabilistic approach to obtain “flat” polynomials [dB08]. This is done through the Salem–Zygmund method (see [K80]), that is, with polynomials of type
\[ \sum_{n=1}^N \varepsilon_n a_n n^{-s} \quad \varepsilon_n = \pm 1 \quad (\text{Rademacher coefficients}). \]
With this and a more refined result than the prime number theorem (a precise estimate on the number of integers smaller or equal to $N$ with prime factors in $[1,y]$), and with similar estimates to those used for Rudin–Shapiro polynomials, one gets the right constant $1/\sqrt{2}$.

**Open problems**

What can be said about the corresponding Sidon constant $S_p(N)$ obtained by replacing $\| \cdot \|_\infty$ by $\| \cdot \|_p$, defined as

$$\|f\|_p = \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T |f(it)|^p dt \right)^{1/p}.$$

For $2 < p < \infty$, by definition of the norms, $S_p(N) \leq S_2(N) = \sqrt{N}$. On the other hand, by Khinchine

$$\mathbb{E}\left( \left\| \sum_{n=1}^N \varepsilon_n a_n n^{-s} \right\|_p \right) \approx \left( \sum_{n=1}^N |a_n|^2 \right)^{p/2}.$$

Choosing $a_n = 1$, $1 \leq n \leq N$, we obtain a Dirichlet polynomial showing that $S_p(N) \geq C\sqrt{N}$. Thus

$$S_p(N) = \sqrt{N} \exp(-O(1)).$$

For $p = 1$, since $\| \cdot \|_1 \leq \| \cdot \|_2$, one has $S_1(N) \geq \sqrt{N}$. Denote by $d(n)$ the number of divisors of $n$. By Cauchy-Schwarz

$$\sum_{n=1}^N |a_n| \leq \left( \sum_{n=1}^N \frac{|a_n|^2}{d(n)} \right)^{1/2} \left( \sum_{n=1}^N d(n) \right)^{1/2}.$$

Results of Bayart and Helson imply that

$$\left( \sum_{n=1}^N \frac{|a_n|^2}{d(n)} \right)^{1/2} \leq \left\| \sum_{n=1}^N a_n n^{-s} \right\|_1.$$

It is also known that $\sum_{n=1}^N d(n)$ grows asymptotically as $\sqrt{N} \sqrt{\log N}$. Thus

$$S_1(N) \leq (1 + o(1)) \sqrt{N \sqrt{\log N}}.$$

As we see, there is a gap between the upper and the lower bounds.

**Conjecture** (Helson).

$$\left\| \sum_{n=1}^N n^{-s} \right\|_1 = o(\sqrt{N}).$$

Of course this would imply that

$$S_1(N) >> \sqrt{N}.$$
REFERENCES


