Geometric representations of graphs

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Touching coin representation

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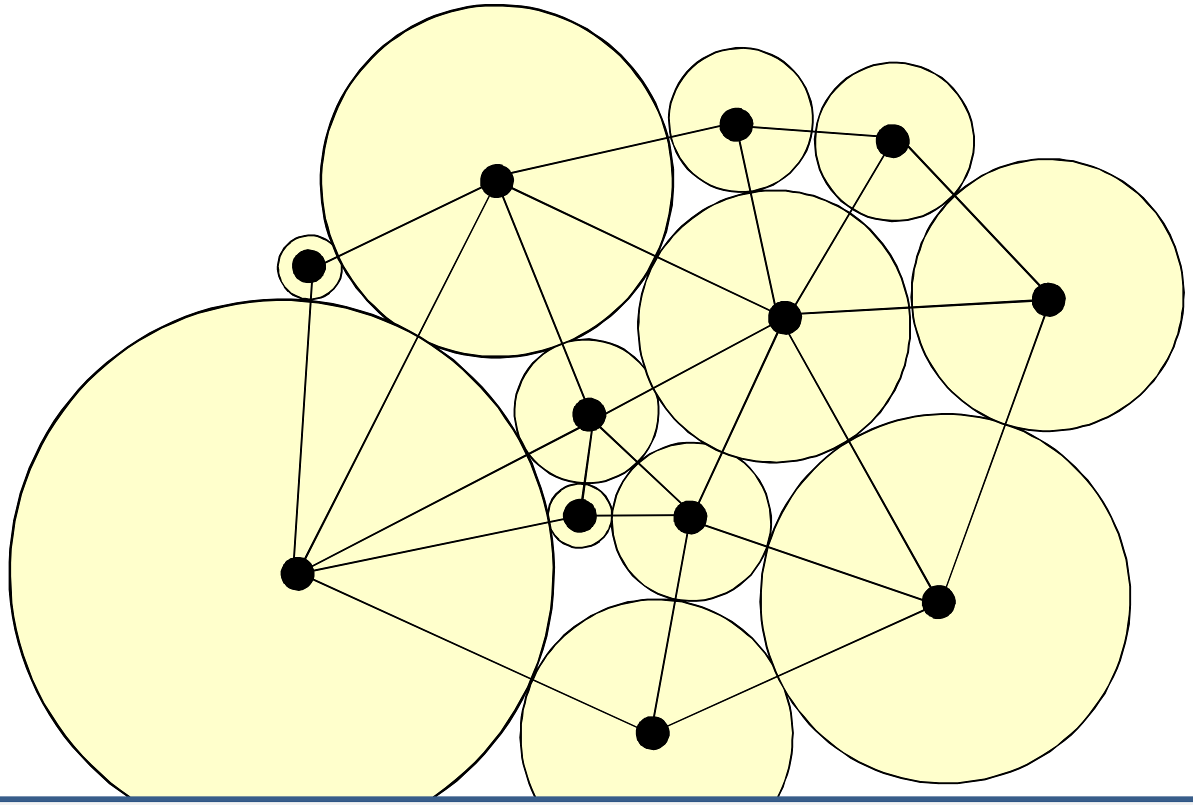
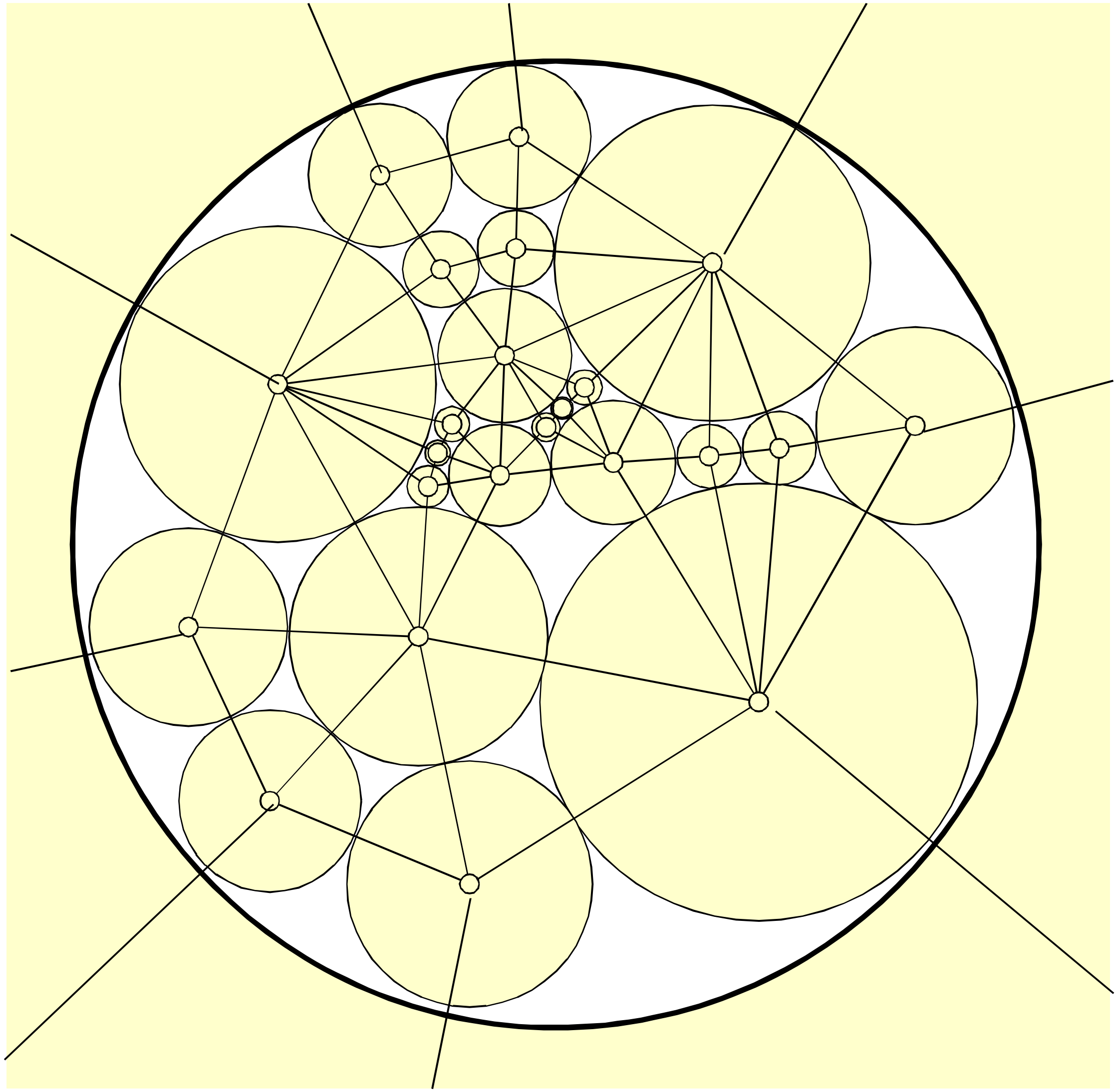


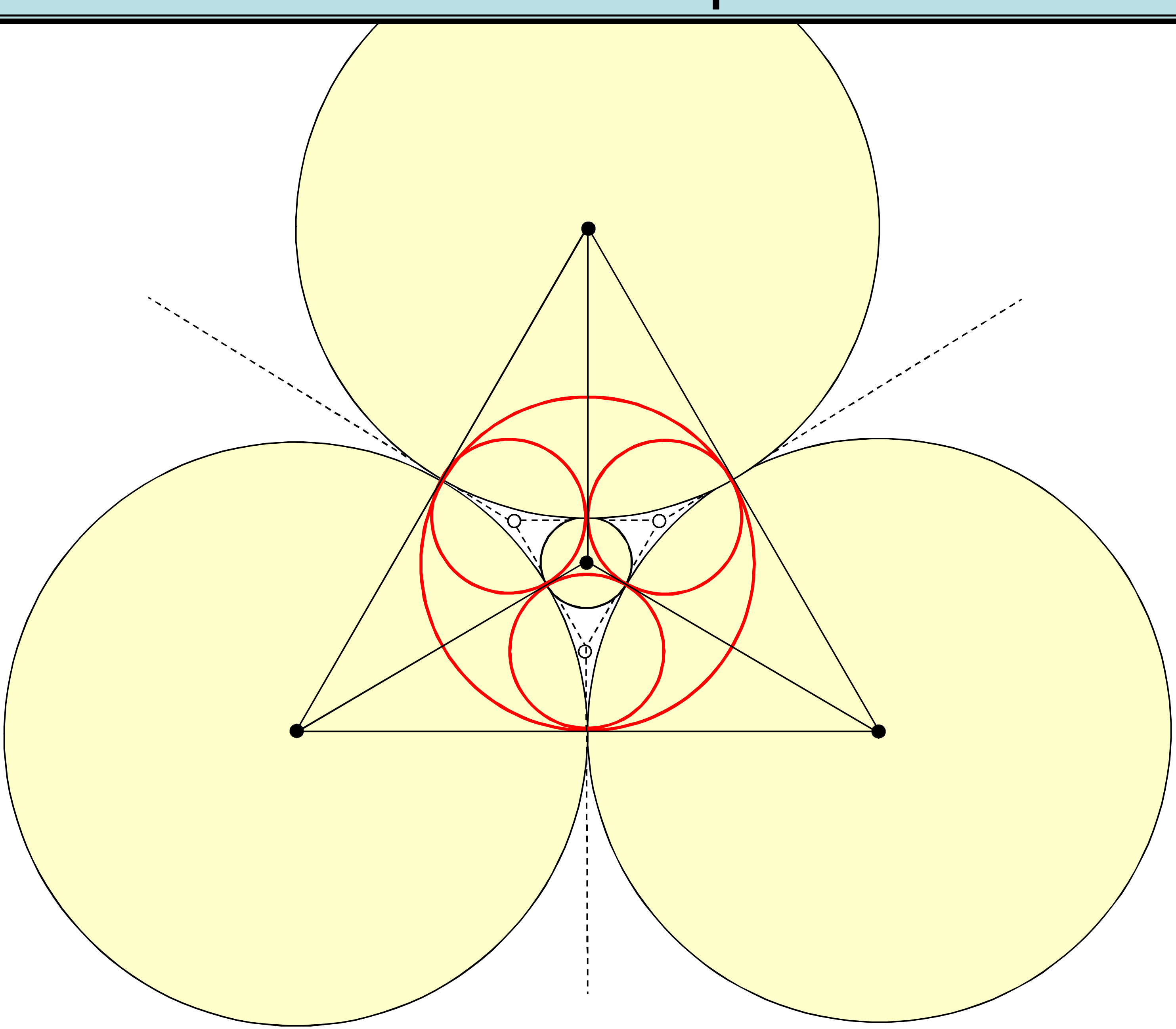
Every simple planar graph can be represented by touching circular disks.

Koebe

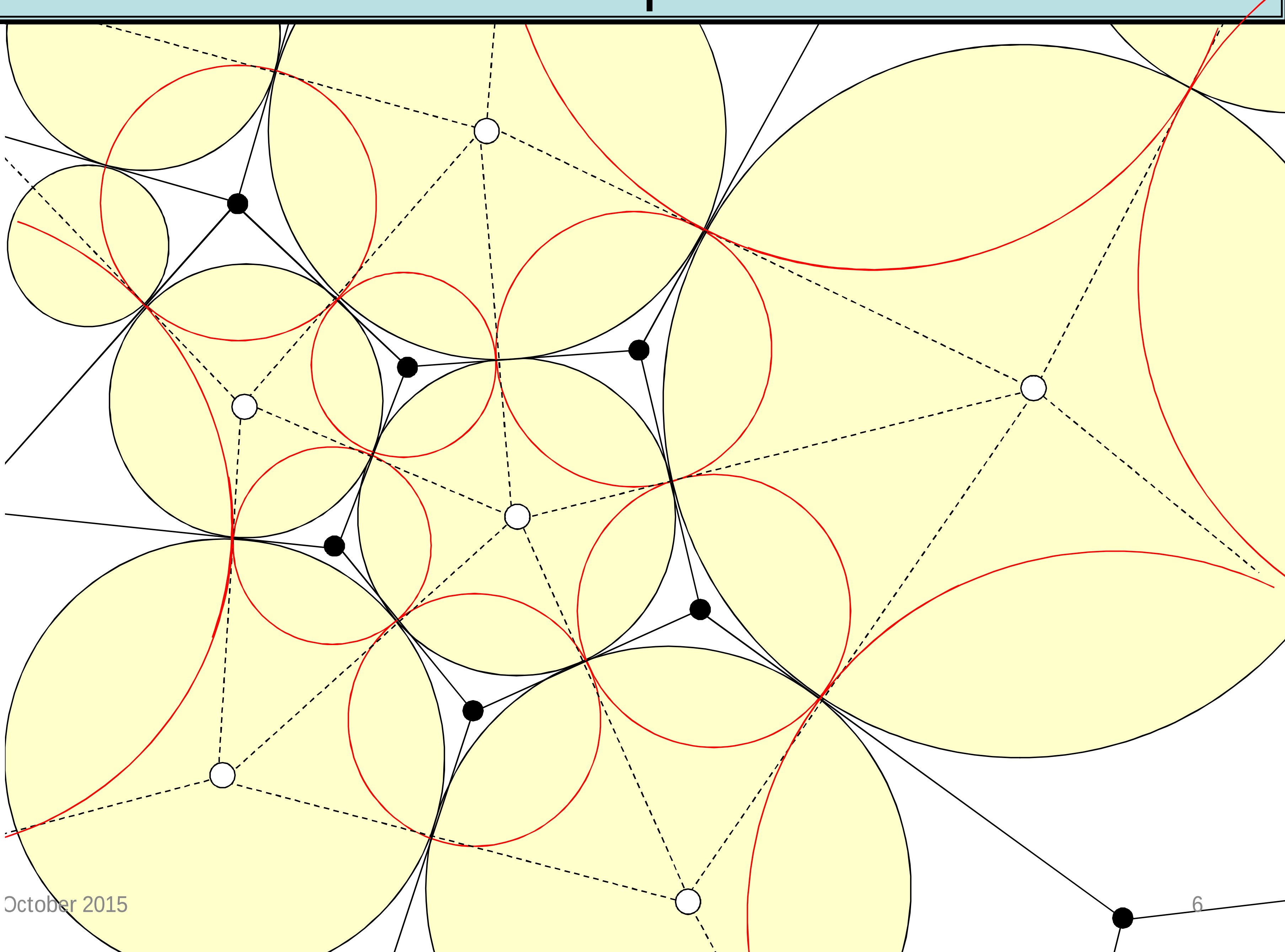
Touching coin representation



Double circle representation



Double circle representation

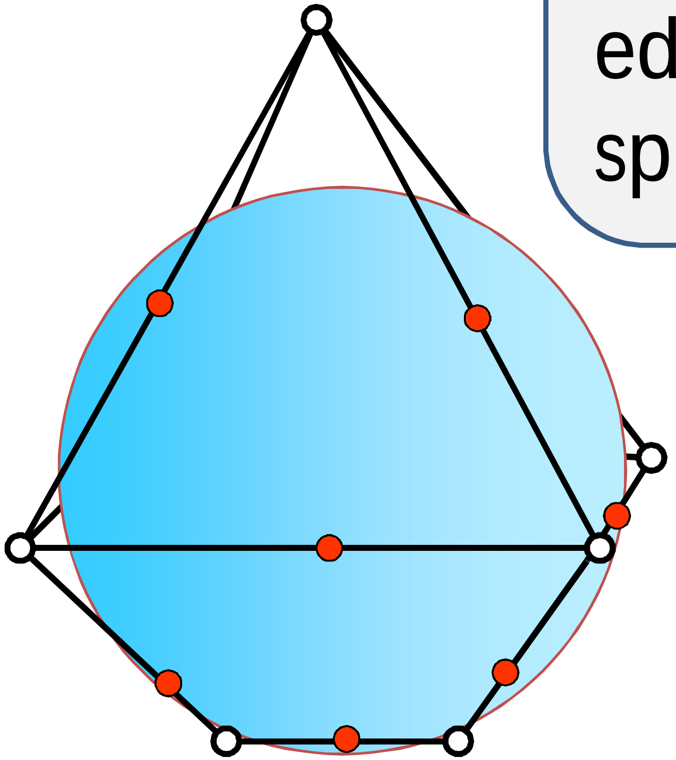


Double circle representation

Every 3-connected simple planar graph has a double circle representation.

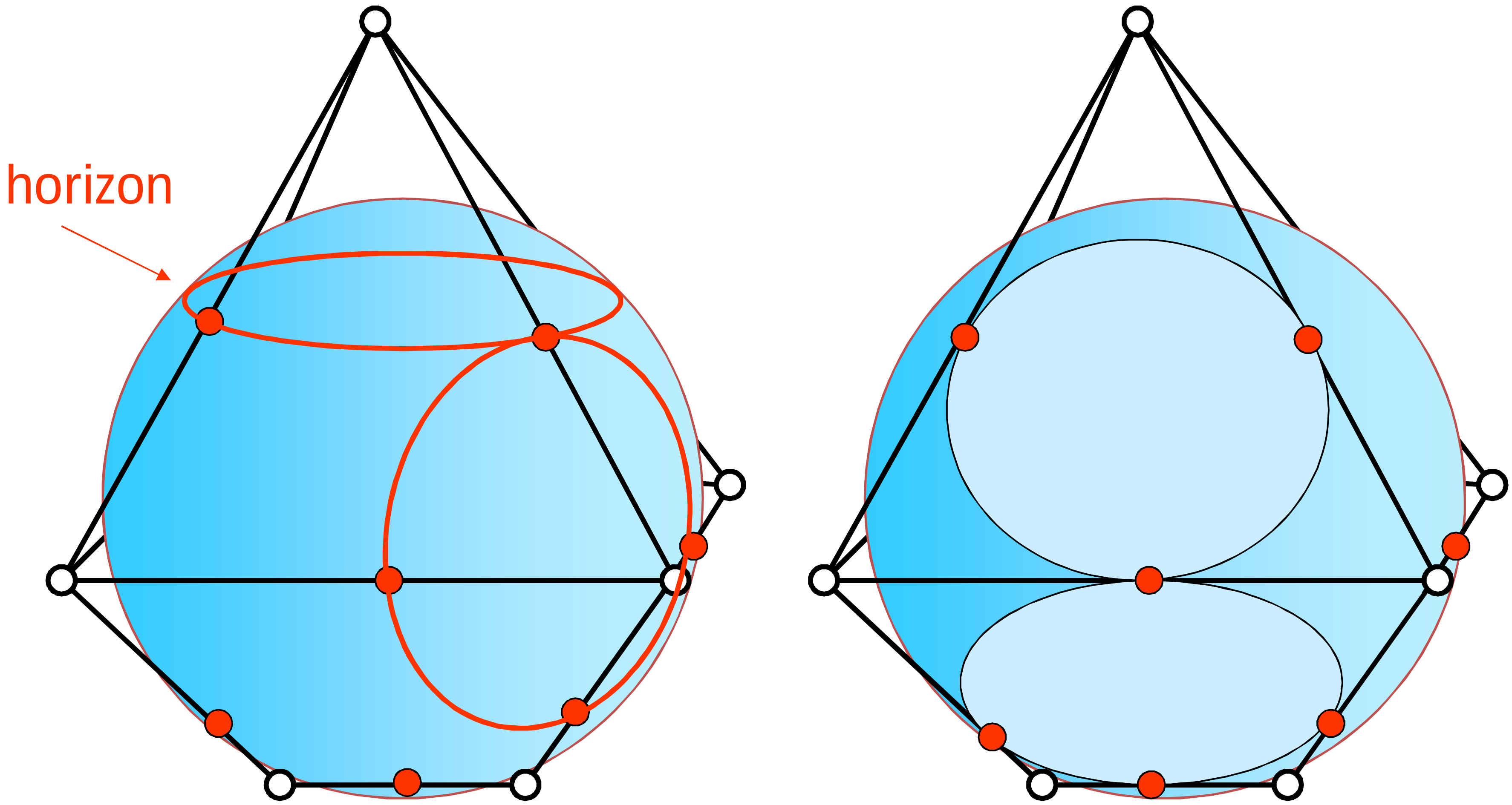
Andre'ev
Thurston

Cage representation

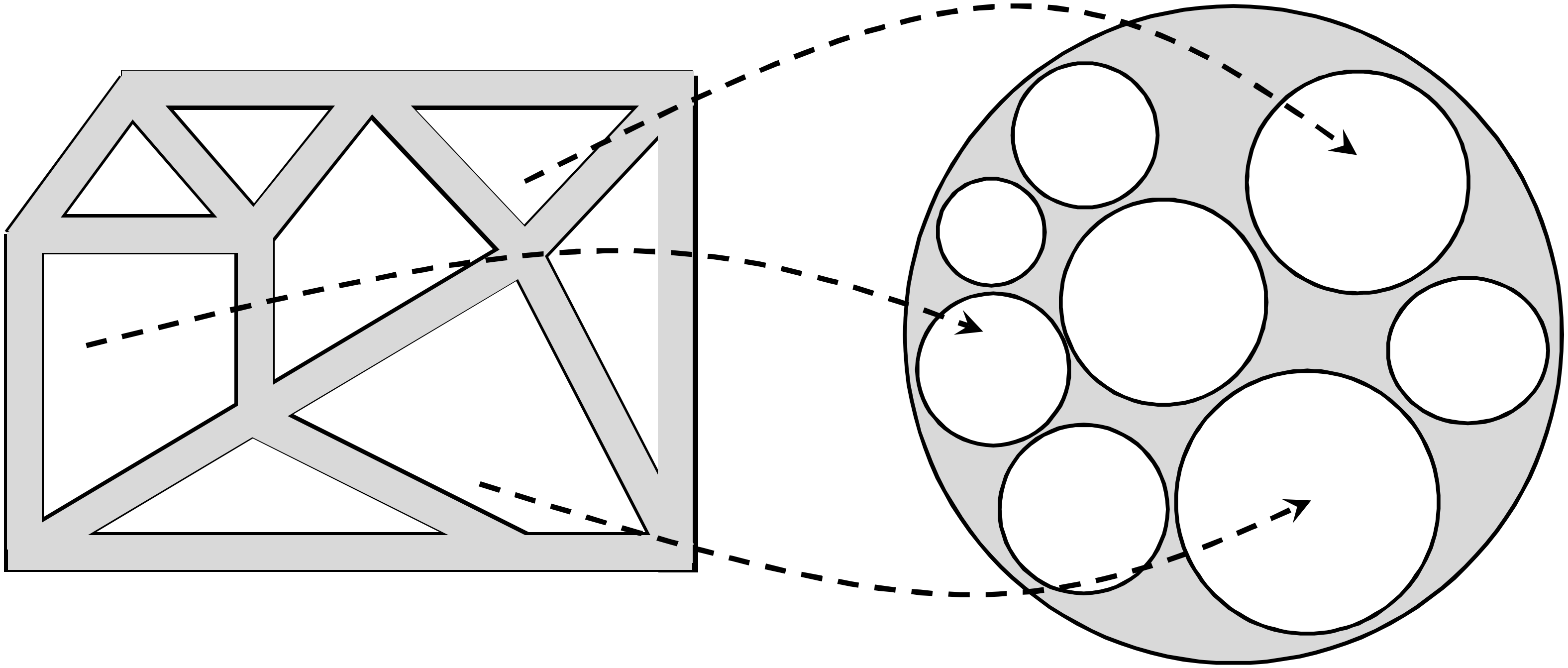


Every 3-connected simple planar graph has representation as the skeleton of a 3-polytope whose edges are tangent to the unit sphere.

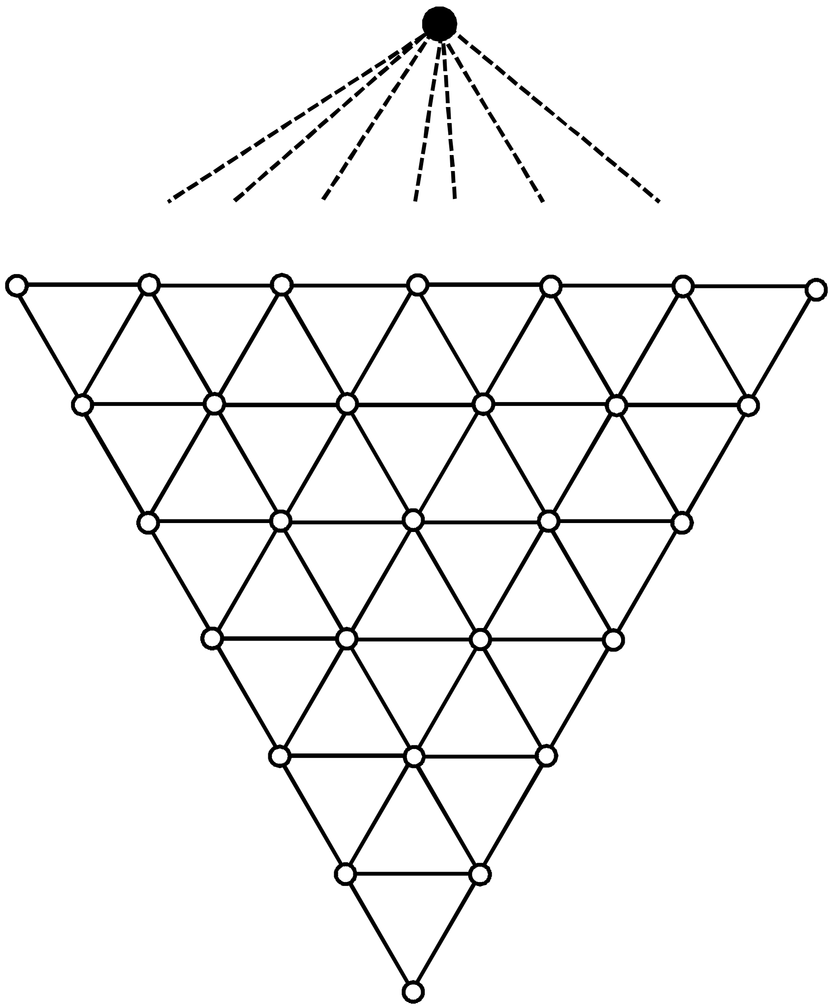
Cage representation

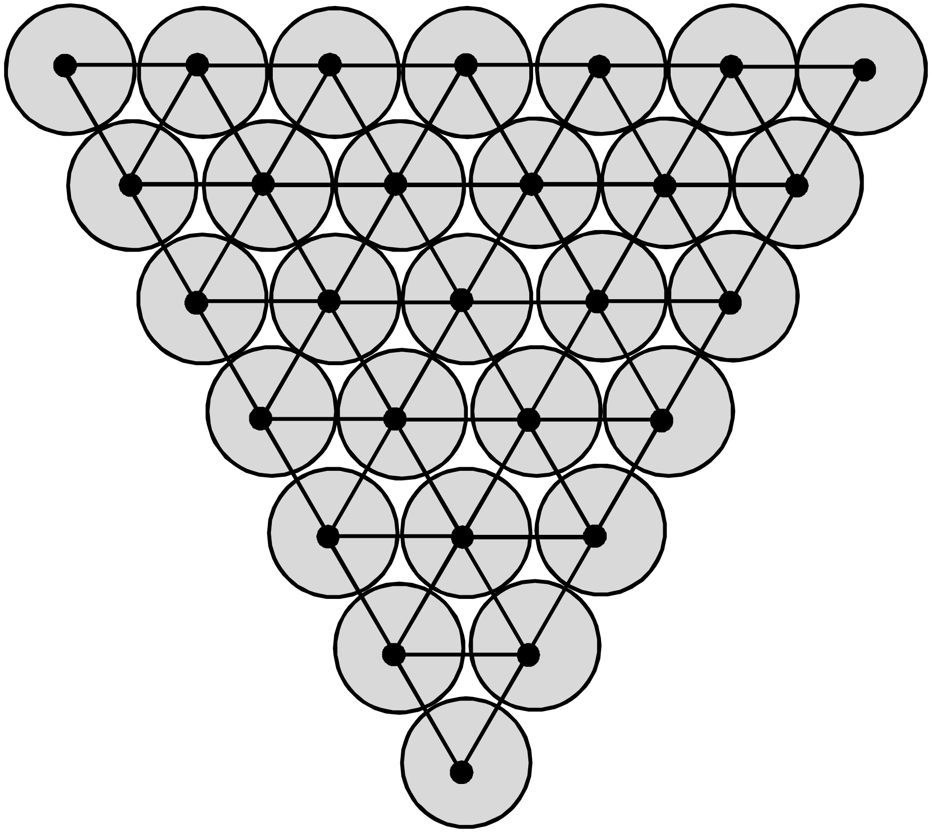


Koebe Circle Domain Thm → touching circle

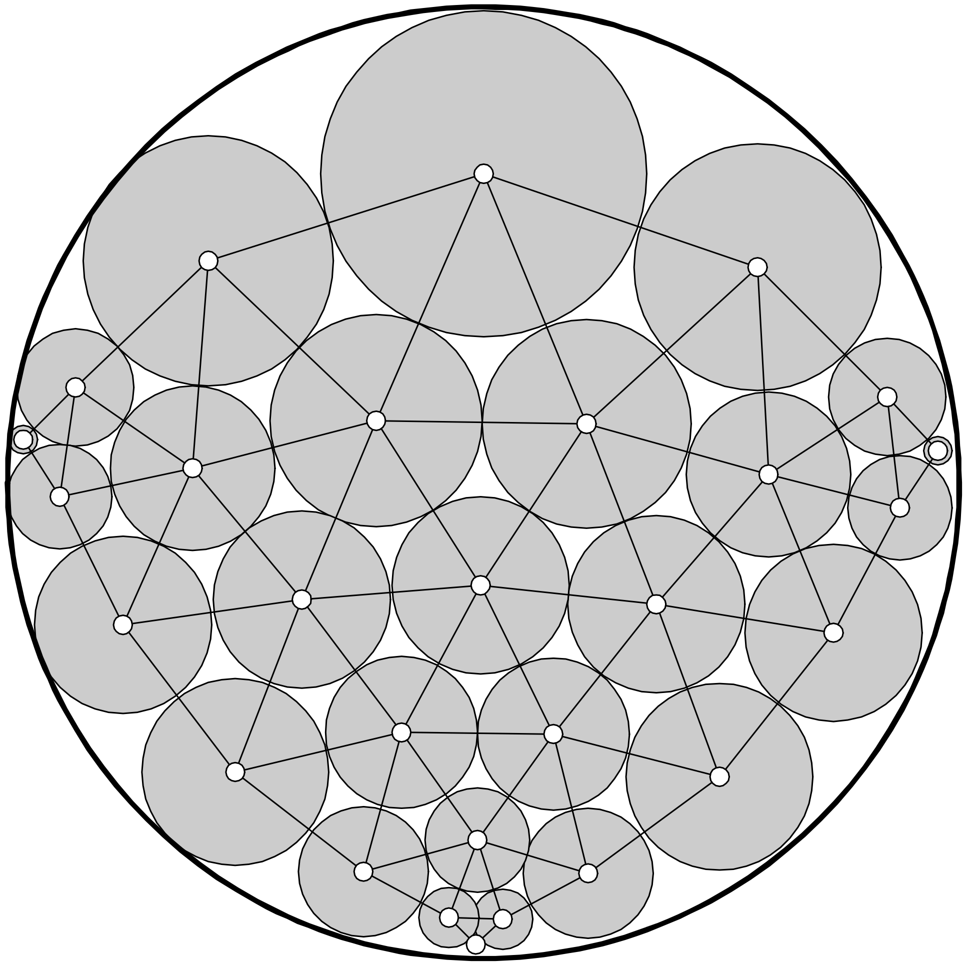


touching circle → ...



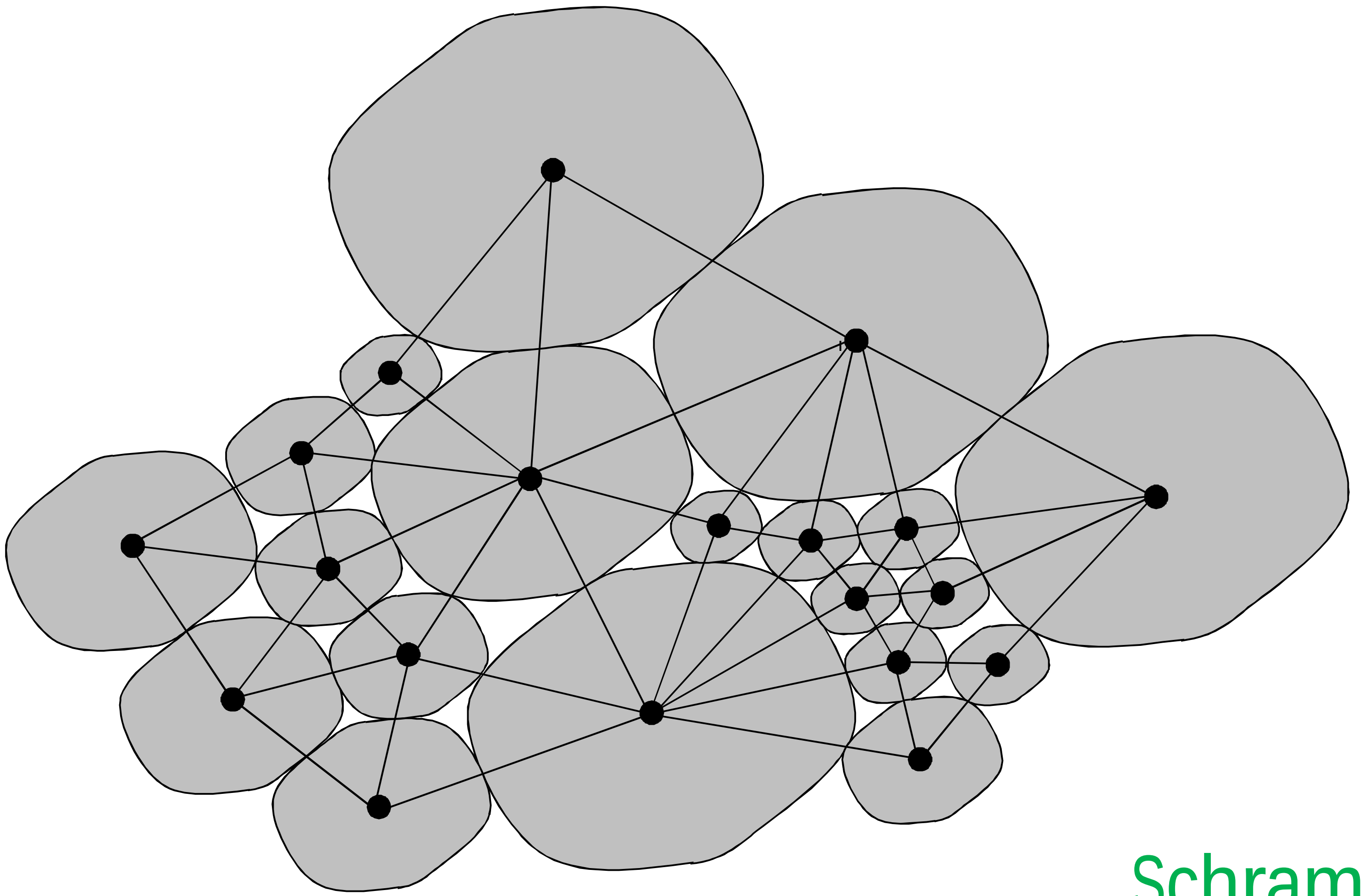


... → Koebe Circle Domain Thm



Rodin and Sullivan

Representation by touching convex domains



Schramm

Algorithmic application

Planar Separator Theorem: ∀planar graph G $\exists S \subseteq V(G)$, $|S| \leq n^{1/2}$ such that ∀connected component of G-S has ≤2n/3 nodes.

Lipton-Tarjan

Algorithm (Miller-Thurston):

- compute coin representation on the sphere
- normalize
- cut by random hyperplane through the origin

2-neighborhood representation

$A=A_G$: adjacency matrix of G

$n=|V(G)|$: number of nodes

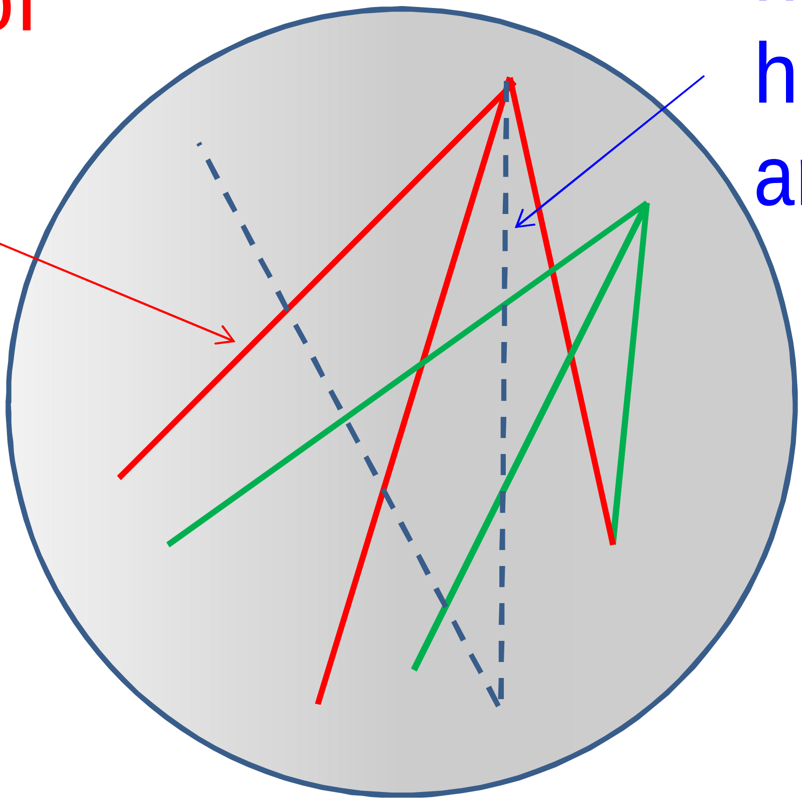
Columns of A: representation in R^n

Columns of A^2: representation in R^n

2-neighborhood representation

probability of connecting = distance

number of 2-paths: highly concentrated around expectation

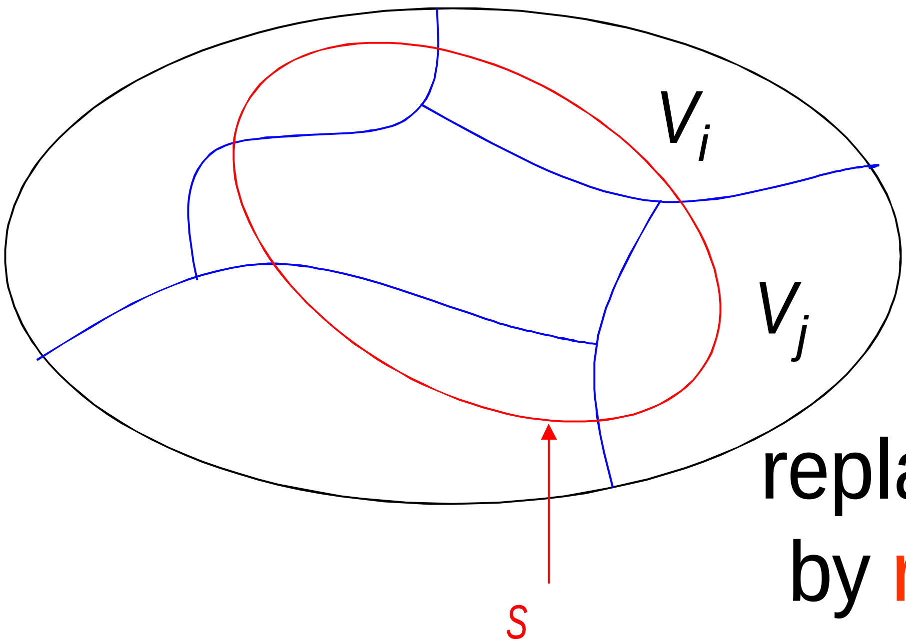


1-neighborhood distance: large

2-neighborhood distance: ~ spherical distance

2-neighborhood representation

Weak Regularity Lemma (Szemerédi)
Frieze-Kannan



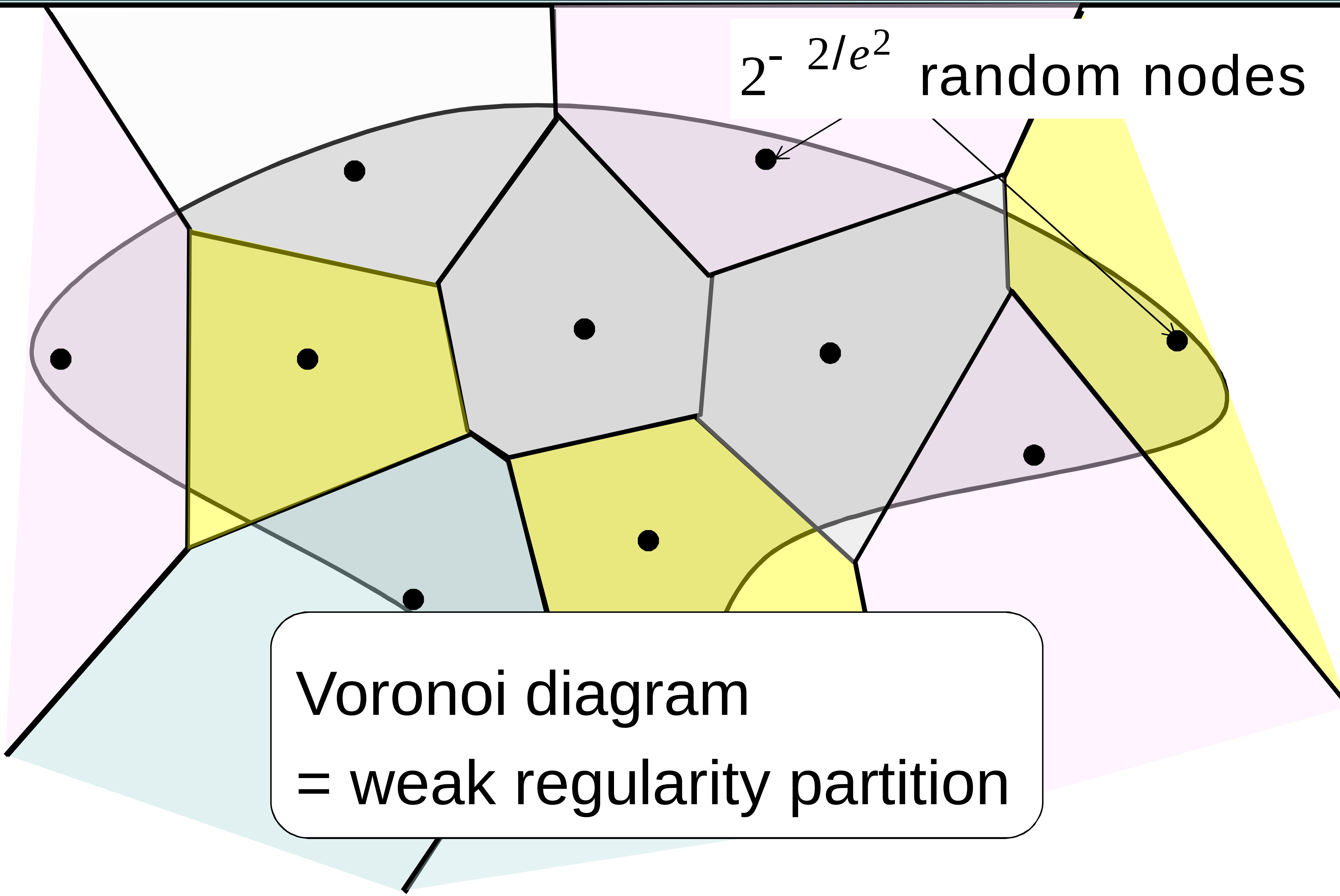
p_{ij} : edge density between V_i and V_j

replace edges between V_i and V_j by random edges with prob p_{ij}

$$\exists \text{ partition } P \text{ with } |P| \le 2^{2/\varepsilon^2}:$$

$$\forall S \subseteq V(G): \ \big|\, |E(G[S])| - |E(G'[S])| \,\big| \le \varepsilon n^2$$

2-neighborhood representation

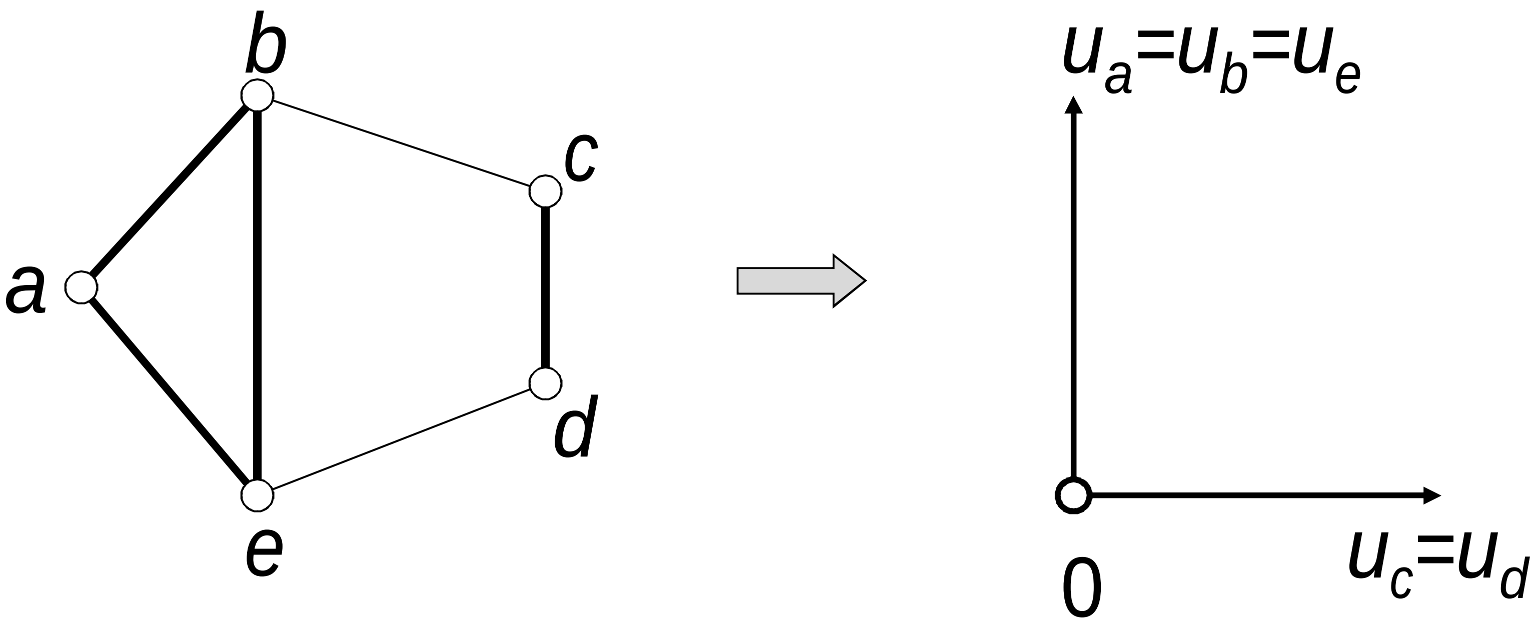


Orthogonal representation

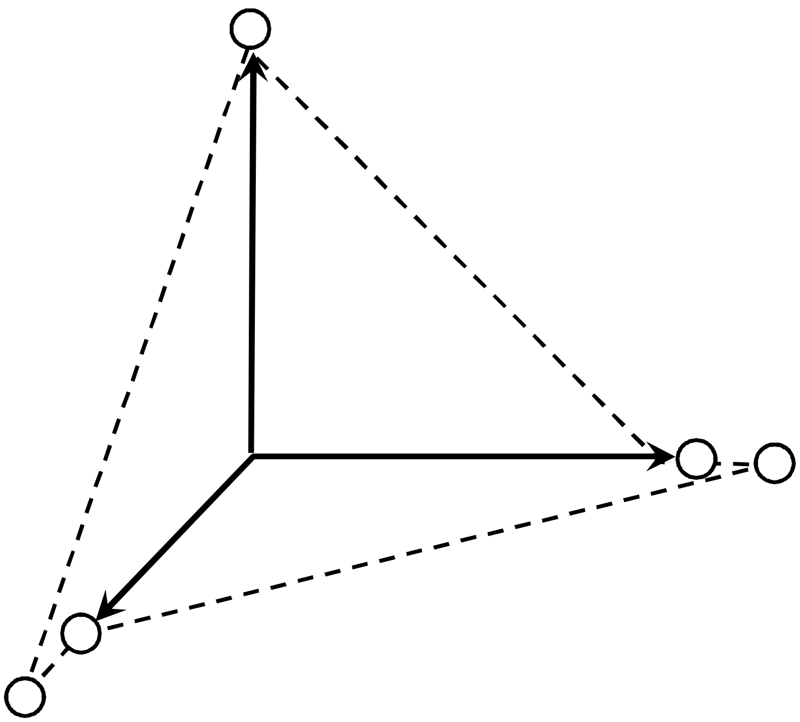
Orthogonal representation

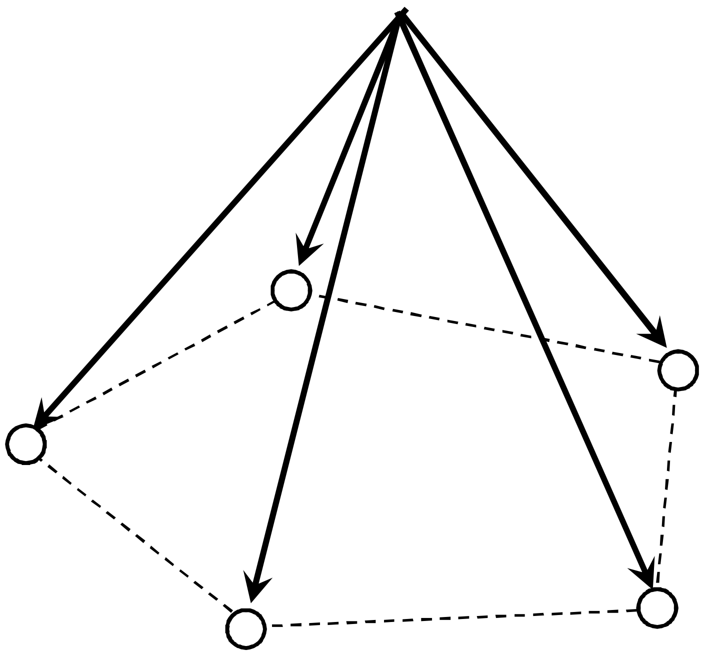
$$i \mapsto u_i \in \mathbb{R}^d \qquad (i \in V)$$

$$u_i \cdot u_j = 0 \qquad (ij \in \overline{E})$$

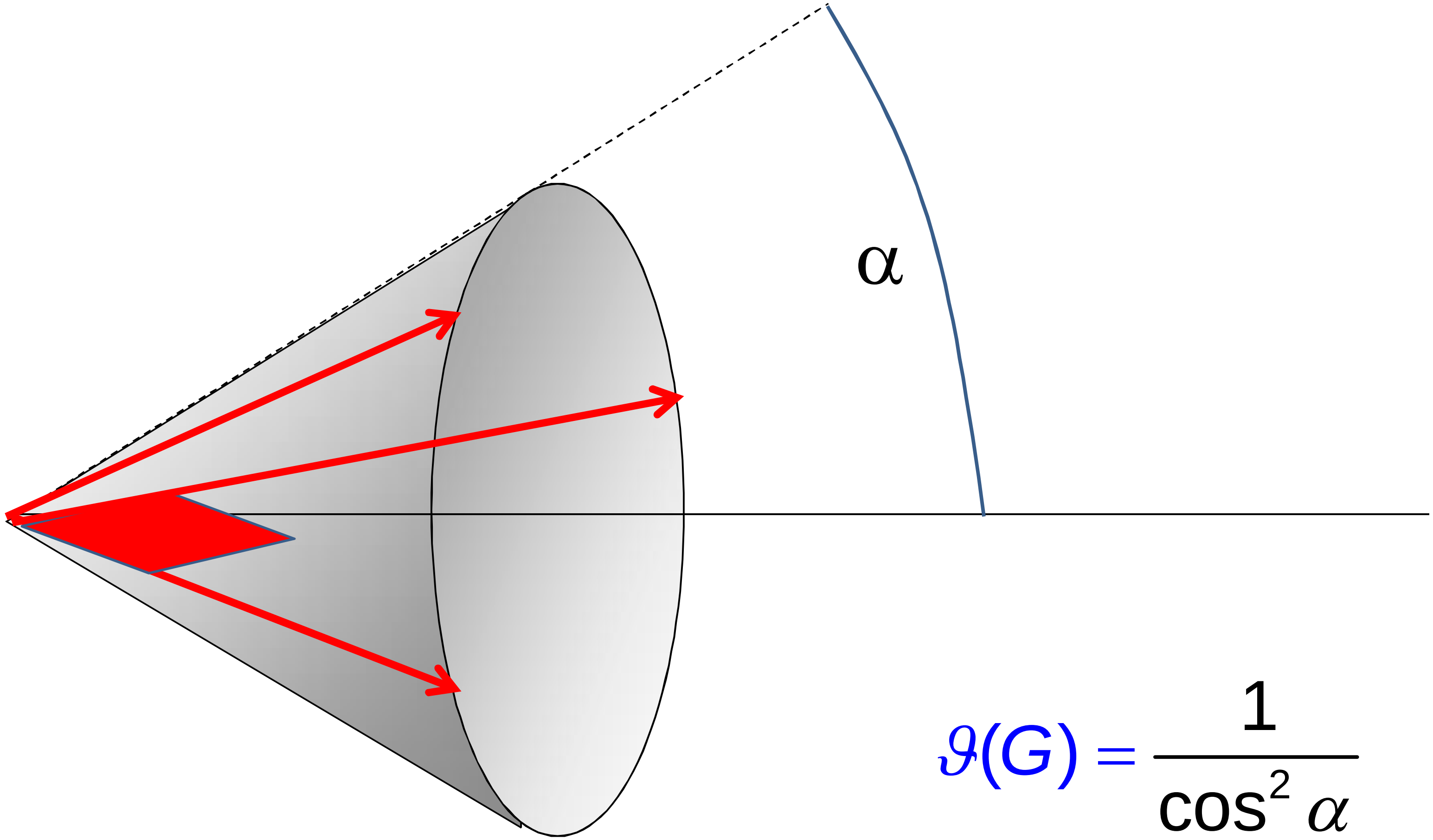


Orthogonal representation





Orthogonal representation



$$\vartheta(G) = \frac{1}{\cos^2 \alpha}$$

max indep set in $G \leq \vartheta(G) \leq$ chromatic number of $\bar{G}$

Orthogonal representation

max indep set in G $\leq \vartheta(G) \leq$ chromatic number of $\bar{G}$

NP-hard

Polynomial time computable

Orthogonal representation in general position

any d linearly independent

G has a general position orthogonal rep in R^d

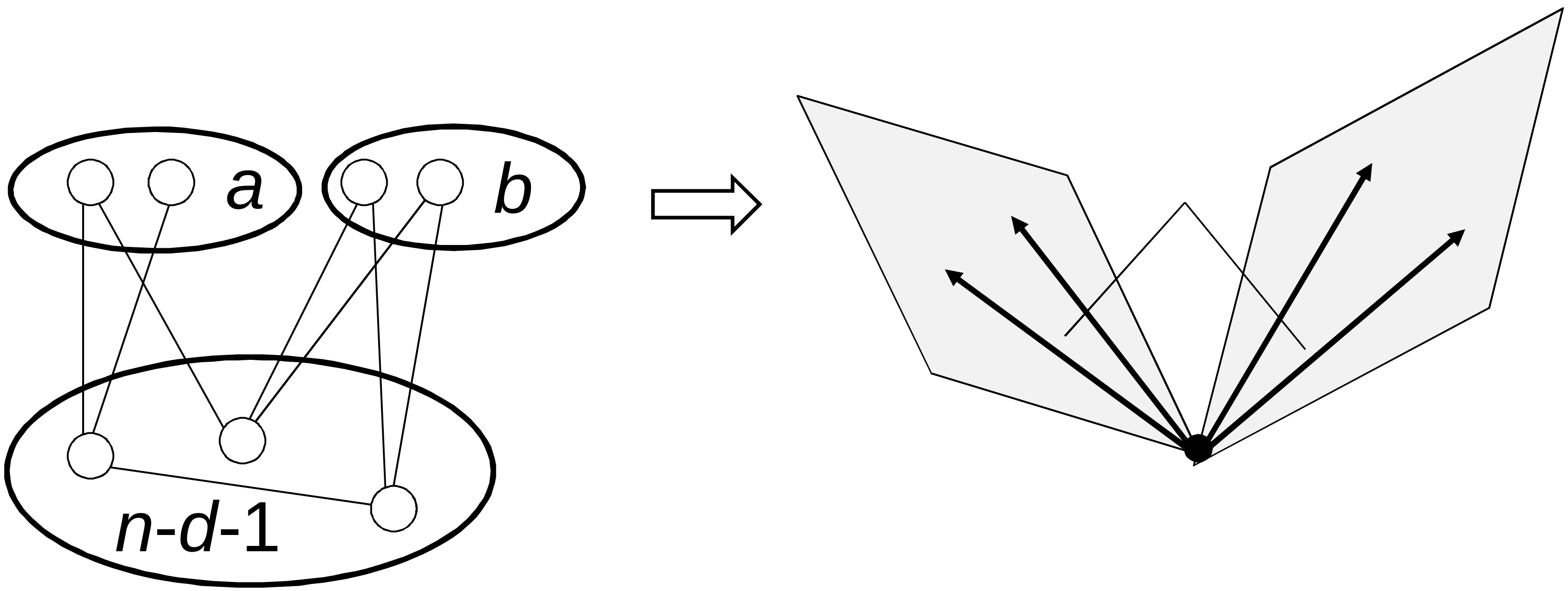
$\Leftrightarrow$

G is $(n\text{-}d)$-connected.

L-Saks-Schrijver

Orthogonal representation in general position

Necessity: not (n-d)-connected $\Rightarrow$ dim > d



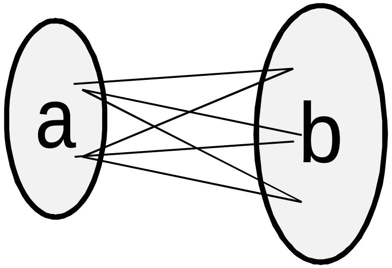
$a+b = d+1$

Variety of orthogonal representations

Think in terms of the complementary graph.

degrees of $\overline{G}$ are $\geq n\text{-}d$ $\Leftrightarrow$ degrees of G are $\leq d$-1

$\overline{G}$ is (n-d)-connected $\Leftrightarrow$ G does not contain $K_{a,b}$ with a+b=d+1



Variety of orthogonal representations

$\mathrm{OR}_d(\overline{G})$: orthogonal reps of $\overline{G}$ in dim d

$$i \mapsto u_i \in \mathbb{R}^d \qquad (i \in V)$$

$$u_i \cdot u_j = 0 \qquad (ij \in E)$$

$\mathrm{DOR}_d(\overline{G})$: degenerate (not in general position) orthogonal reps of $\overline{G}$ in dim d

$$\prod_{1 \le i_1 < \ldots < i_d \le n} \det(u_{i_1}, \ldots, u_{i_d}) = 0.$$

} algebraic varieties

Variety of orthogonal representations

all degrees of G are $\leq d$-1

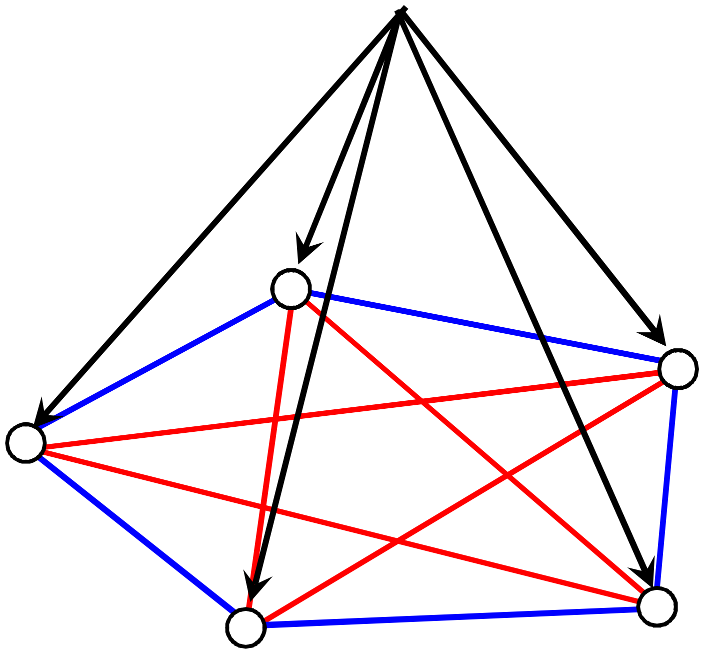
$$\Rightarrow$$

$\mathrm{OR}_d(\overline{G}) \setminus \mathrm{DOR}_d(\overline{G})$ is irreducible.

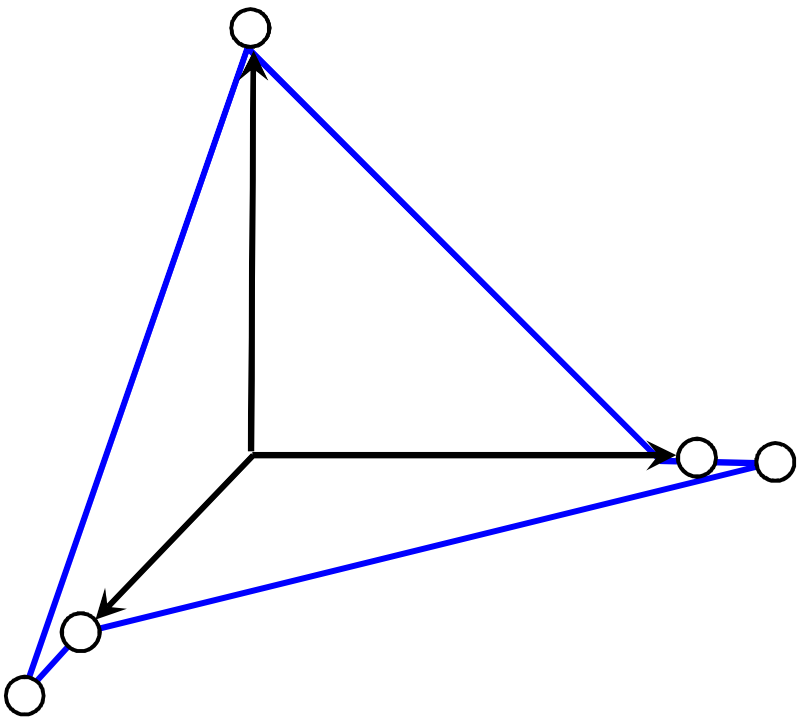
L-Saks-Schrijver

What about $\mathrm{OR}_d(\overline{G})$*?*

Variety of orthogonal representations



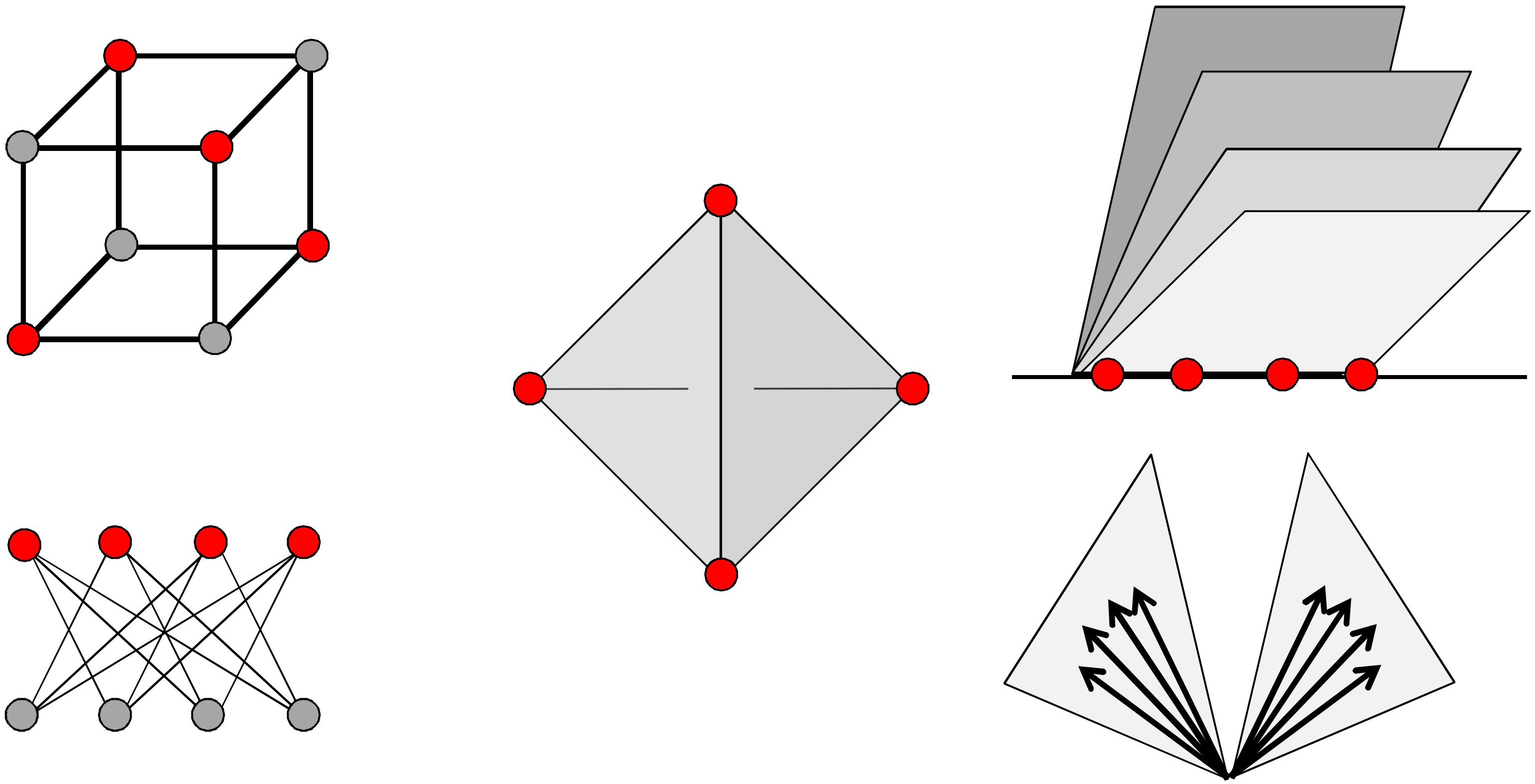
non-degenerate



degenerate

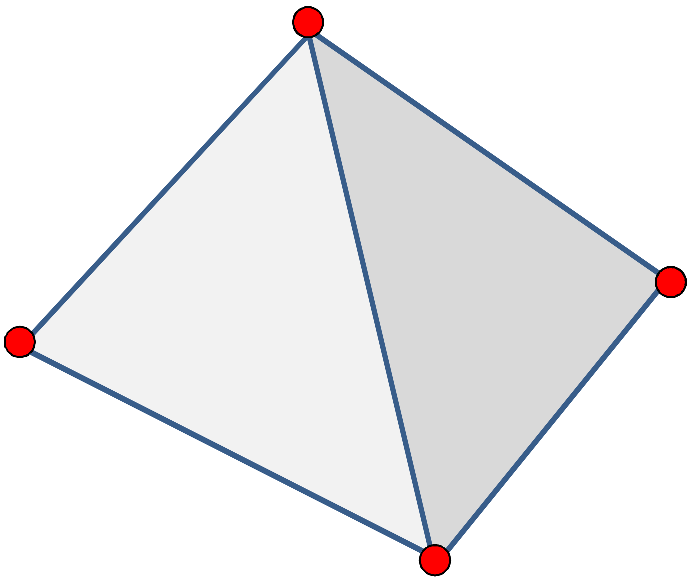
but limit of non-degenerate ones

Variety of orthogonal representations

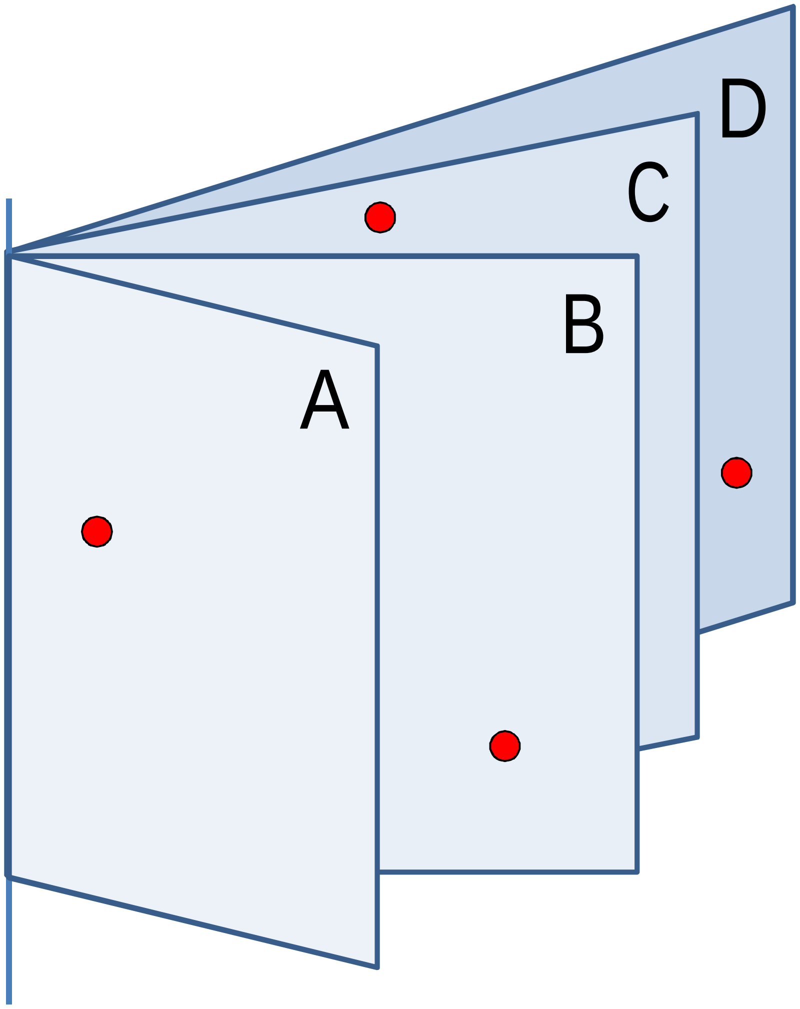


non-degenerate   degenerate

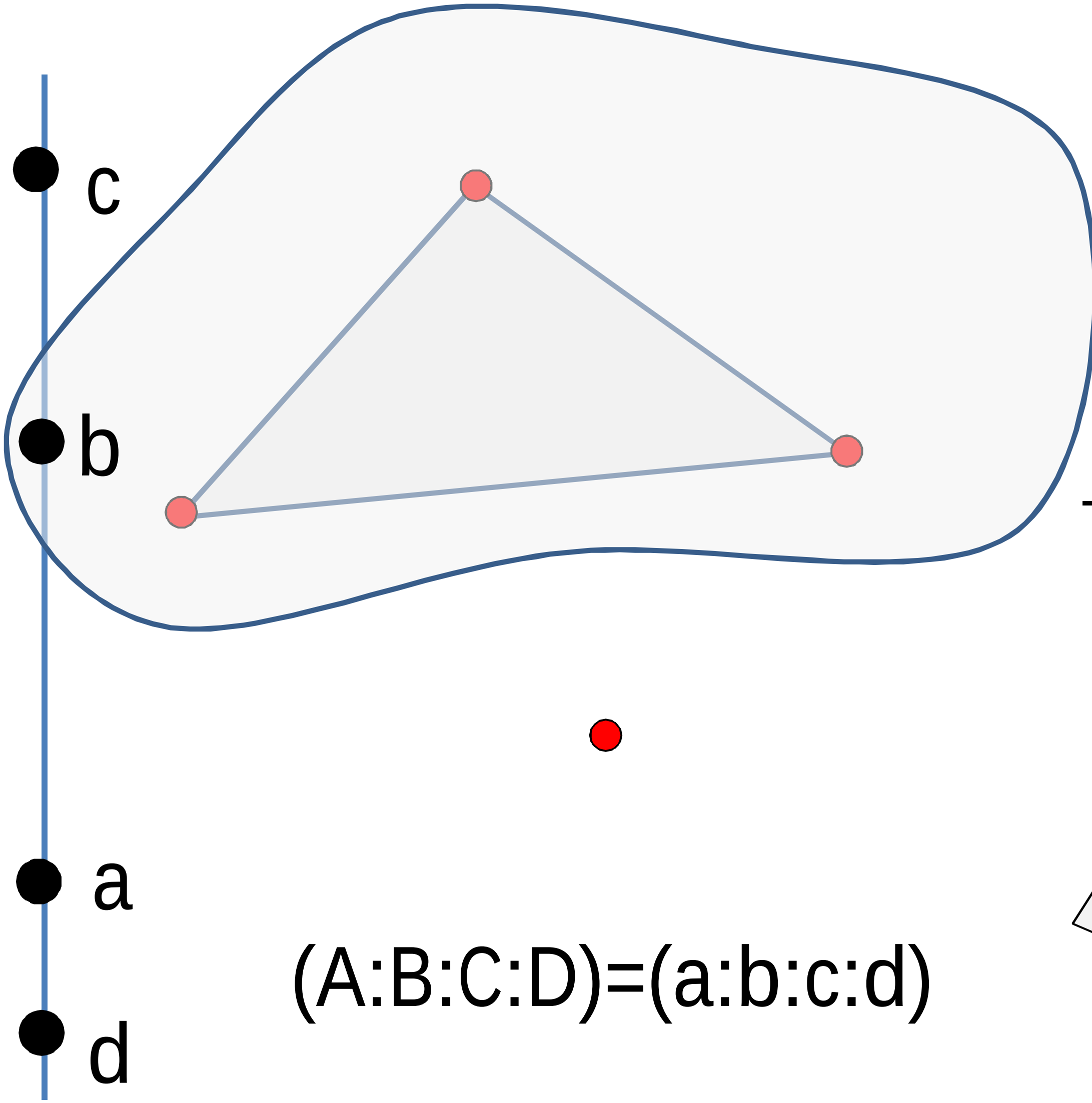
Variety of orthogonal representations

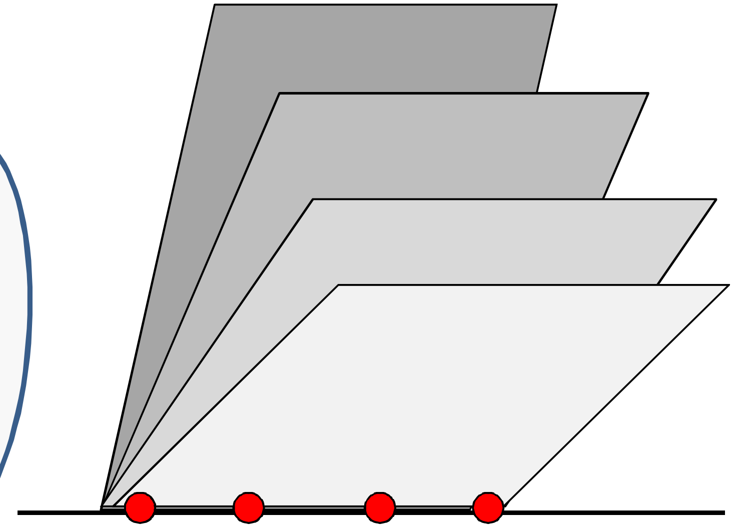


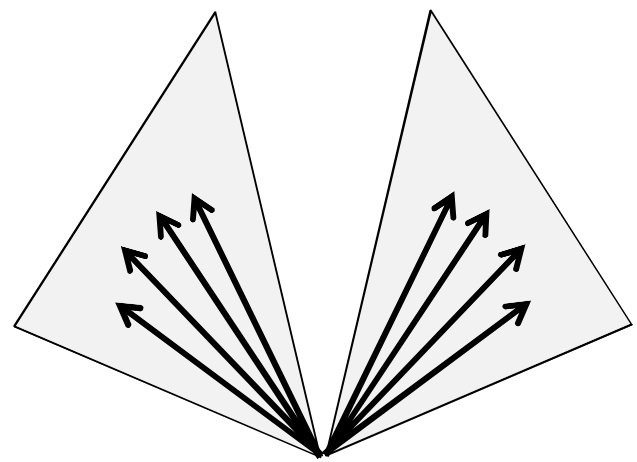
Variety of orthogonal representations



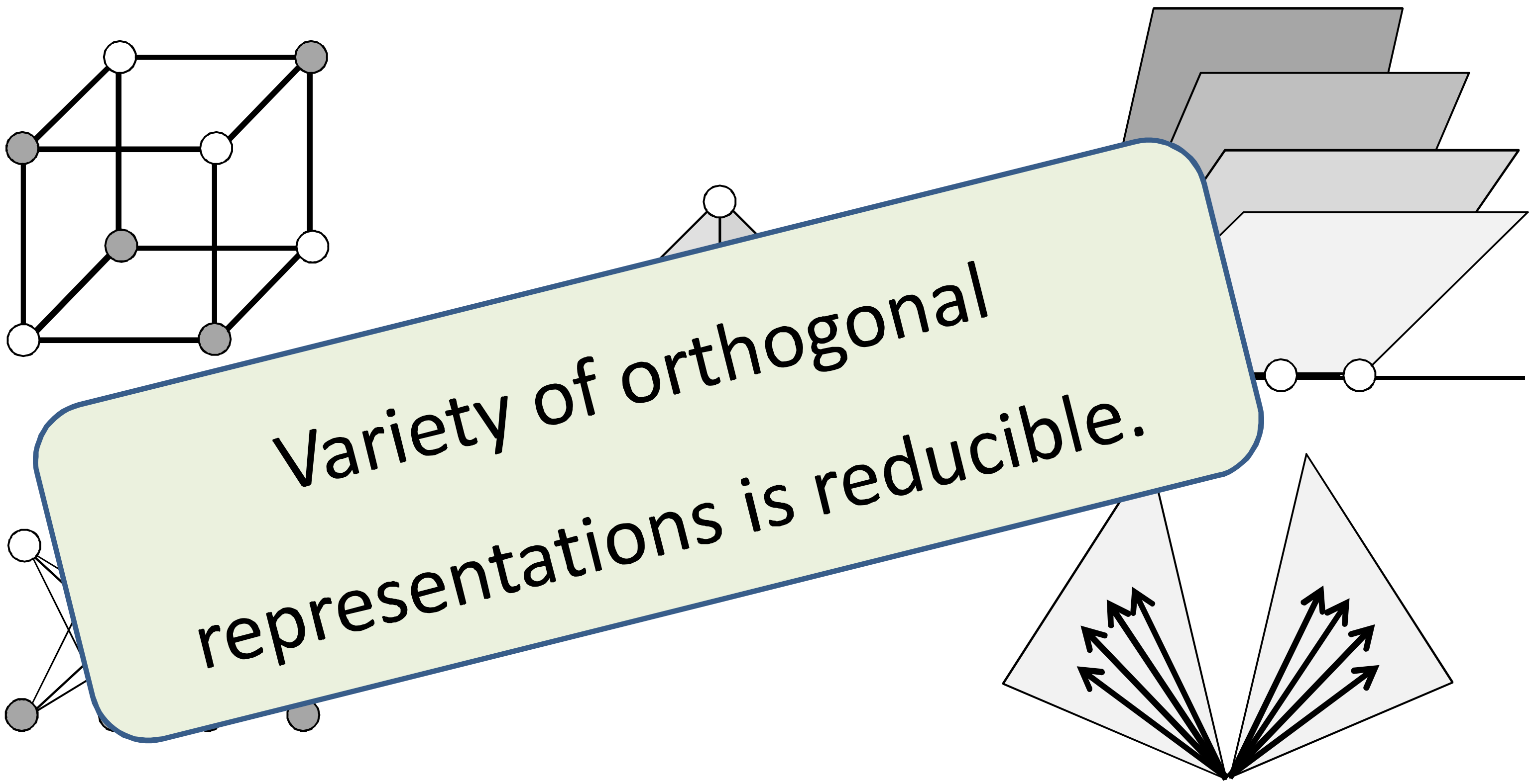
Variety of orthogonal representations







Variety of orthogonal representations



Variety of orthogonal representations is reducible.

non-degenerate degenerate

Variety of orthogonal representations

Assume all degrees of G are $\leq d$-1. Then $\mathrm{OR}_d(\overline{G})$ is irreducible

$$\Leftrightarrow$$

orthogonal representations in general position are dense in $\mathrm{OR}_d(\overline{G})$.

L

Variety of orthogonal representations

Assume G does not contain $K_{a,b}$ with $a+b=d+1$.
$\mathrm{OR}_d(\overline{G})$ is irreducible for

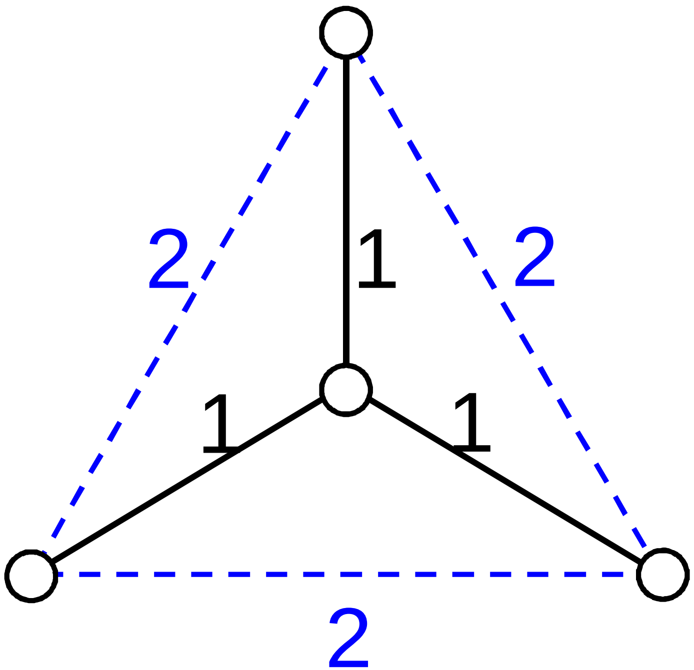
(a) for $d \leq 3$

(b) for $d=4$ except if G is the cube.

L

Distance representation

Distance representation



Graph distances are not representable in euclidean space!

How to construct approximate representations?

Distance representation

$d(u,v)$: distance of nodes u and v in G.

S_k: random subset of V with $P(i \in S_k) = 2^{-k}$

i □ $x(u) = (d(i,S_1), d(i,S_2),\ldots, d(i,S_m))$

$m \approx \log n$

Distance representation

With high probability $\forall$u, v

$$d(u,v) \leq |x(u)-x(v)| \leq m\, d(u,v).$$

Bourgain

Applications:

Works for any finite metric space

- Leighton-Ra
  MFMC Theorem for multicommodity flows

  Linial – London – Rabinovich

- Approximate bandwidth Feige

Distance representation is volume respecting.

Feige

With high probability $\forall$nodes $u_1,\ldots,u_m$

$$\frac{\mathrm{vol}(u_1,\ldots,u_m)}{O(\log^5 n)} \le \mathrm{vol}(\mathrm{conv}(x(u_1),\ldots,x(u_m))) \le \mathrm{vol}(u_1,\ldots,u_m)$$

What is this?

$$\mathrm{vol}(u_1,\ldots,u_m) = \frac{1}{(m-1)!} \min_{\substack{T \text{ tree on} \\ \{u_1,\ldots,u_m\}}} \prod_{u_i u_j \in E(T)} d(u_i,u_j)$$

Project on a random line!